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GEOMETRY OF WEAKLY SELF-DUAL KÄHLER SURFACES

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Introduction

In this paper we present the local study of weakly self-dual Kähler surfaces, *i.e.* whose anti-self-dual Weyl tensor is harmonic. The local classification of these surfaces has been done in [ACG03] by V. Apostolov, D. Calderbank and P. Gauduchon, where it is given an explicit local description of these metrics.

The weakly self-dual Kähler surfaces are a generalization of self-dual Kähler surfaces, which have been considered in several recent works. They appear as a particular case (for $n = 4$) of the study done by R. Bryant in [Br01] on Bochner-flat Kähler manifolds, as well as in the paper [AG02] of V. Apostolov and P. Gauduchon, where an equivalence has been established between self-dual Kähler surfaces and self-dual Hermitian Einstein metrics and where an explicit local description of the latter is provided.

In the above mentioned papers there has been proved that a self-dual Kähler surface is bi-extremal and this fact has been extended to the case of weakly self-dual Kähler surfaces in [ACG03], starting from the Matsumoto-Tanno identity for these surfaces. On the other hand, in contrast to the self-dual Kähler metrics, where the compact examples are all locally symmetric, there has been observed, also in [ACG03], that in Calabi's family of extremal Kähler metrics, on the first Hirzebruch surface F_1, there is a unique weakly self-dual metric up to homothety.

The main result in the classification of weakly self-dual Kähler surfaces is the following:

Teorema 0.1. ([ACG03]) *Let (M, g, J, ω) be a weakly self-dual Kähler surface. Then (g, J) is a bi-extremal Kähler metric in the sense that the scalar curvature and the Pfaffian of the normalized Ricci form of (g, J) are Poisson-commuting momentum maps for Hamiltonian Killing vector fields K_1 and K_2 respectively. Furthermore, on each connected component of M one of the following holds:*

(i) *K_1 and K_2 are linearly independent on a dense open set. Then (g, J, ω) has an explicit local form depending on an arbitrary polynomial of degree 4 and an arbitrary constant which is zero if and only if g is self-dual.*

(ii) *K_1 is non-vanishing on a dense open set, but $K_1 \wedge K_2$ is identically zero. Then (g, J, ω) is locally of cohomogeneity one and is given explicitly by the Calabi construction.*

(iii) *K_1 and K_2 vanish identically. Then g has parallel Ricci curvature, hence is either Kähler–Einstein or locally a Kähler product of two Riemann surfaces of constant curvatures.*

If (M, g, J, ω) is compact and connected, then it necessarily belongs to case (ii) *or* (iii) *above, and in case* (ii) *(M, g, J, ω) is isomorphic to the weakly self-dual Calabi extremal metric on F_1 (*cf. *Theorem* 5 *in* [ACG03]*).*

An important aspect of the aproach of weakly self-dual Kähler surfaces is their study within a more general setting. The Matsumoto-Tanno identity for weakly self-dual Kähler surfaces is equivalent to the fact that the primitive part ρ_0 of the Ricci form satisfies an overdetermined linear differential equation. On the open set where ρ_0 doesn't vanish, the equation means that ρ_0 defines a conformally Kähler Hermitian structure $((|\rho_0| / \sqrt{2})^{-2} g, I)$ to the given metric g and which induces the opposite orientation to J. Many of the properties of weakly self-dual Kähler surfaces are consequences of the fact that ρ is a closed J-invariant 2-form, whose primitive part satisfies this equation. In Theorem 4.1 it is proved that two of the algebraic invariants (the trace and the Pfaffian) of any such 2-form are Poisson-commuting momentum maps for Hamiltonian Killing vector fields. Therefore, we first study in general the theory of Kähler surfaces with such 2-forms, called Hamiltonian, which include as a particular case the weakly self-dual ones. We mention that a part of these results have been generalized in [ACG03] for the case of almost Kähler surfaces (M, g, J, ω) with J-invariant Ricci tensor, thus providing new non-compact counterexamples to the still open Goldberg conjecture.

The paper is organized as follows. In a first introductory section we considered important to briefly present the steps followed in order to classify the weakly self-dual Kähler surfaces, which could give the reader an overview of the classification. The second section contains the notations and conventions used throughout, as well as some basic results necessary for the proves.

The next sections contain the presentation of the local classification. In a first part we give the definition of weakly self-dual Kähler surfaces and through equivalent conditions we get to naturally consider the notion of a Hamiltonian 2-form. Thus, in the following section we study the properties of surfaces admitting a Hamiltonian 2-form. Afterwards, we apply these general results in the particular case of weakly self-dual Kähler surfaces, which are proved to be bi-extremal and we get a rough classification of these surfaces.

In the last section we present the explicit local description of weakly self-dual Kähler surfaces. The first case is the one in which the Hamiltonian Killing vector fields, whose existence is given by the bi-extremal structure, are linearly independent, meaning that the Kähler structure (g, J, ω) is toric. Theorem 6.1 characterizes the class of Kähler toric structures arising in this way from a Hamiltonian 2-form. While toric

Kähler surfaces depend in general on an arbitrary function of two variables ([Ab98],[Gui94]), the toric surfaces which come from Hamiltonian 2-forms, called „ortho-toric" ([ACG03]), have an explicit form given by Proposition 6.2, depending on two arbitrary functions of one variable. This fact has the great advantage that the differential equations coming from the conditions imposed on curvature are ordinary differential equations. In particular, we obtain explicitly all the extremal toric Kähler structures coming from Hamiltonian 2-forms, including new examples of Kähler metrics that are conformally Einstein, but neither self-dual nor anti-self-dual, amd also some explicit Einstein metrics. The weakly self-dual Kähler metrics in this family are classified in Theorem 6.2. The second case, in which the Hamiltonian Killing vector fields are linearly dependent, but not both zero, is related to the Calabi construction of Kähler metrics on line bundles over a Riemann surface ([Cal82]). In Theorem 6.4 is obtained the classification of weakly self-dual Calabi extremal metrics: a four parameter family, one of which is globally defined on the first Hirzebruch surface, F_1.

In the two appendices of the paper we present the decomposition of the curvature tensor and the extremal Kähler metrics introduced by Calabi in [Cal82], in order to justify the definition given in the paper.

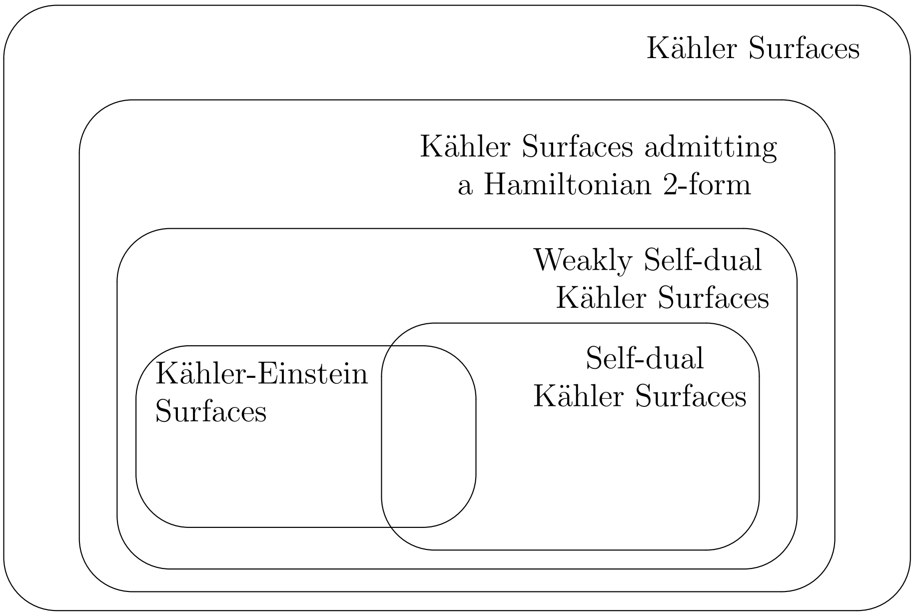
1. An Overview of the Classification of Weakly Self-dual Kähler Surfaces

In this first section we briefly present the main steps of the classification of weakly self-dual Kähler surfaces in order to give an overview of the content of this paper.

A Kähler surface is called weakly self-dual if its anti-self-dual Weyl tensor is harmonic.

Possible framings of weakly self-dual Kähler surfaces in the general study of Kähler surfaces are the following:

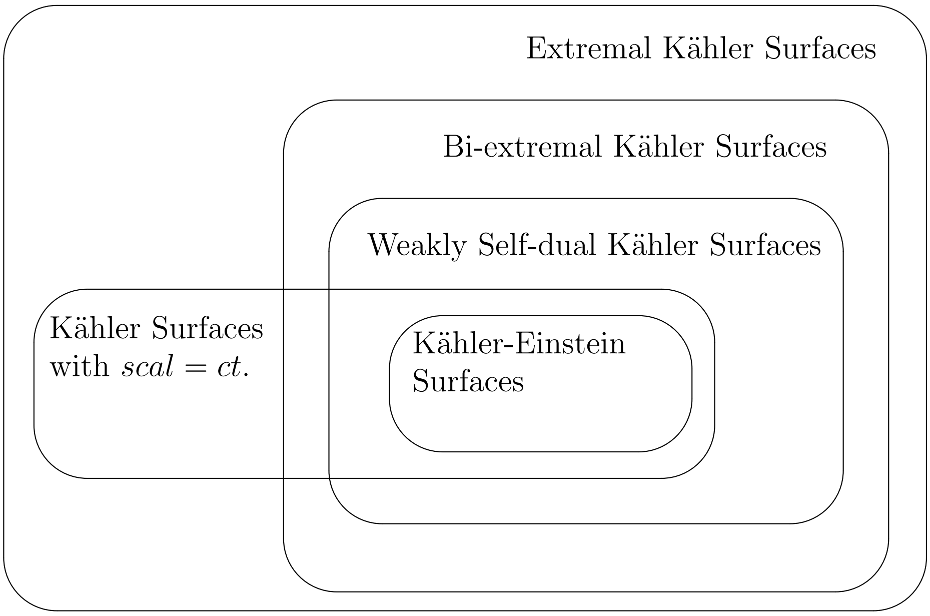
1) The weakly self-dual Kähler surfaces are a generalization of the self-dual Kähler surfaces (characterized by the fact that the anti-self-dual Weyl tensor vanishes) and also a generalization of Kähler-Einstein surfaces[1]. On the other hand it will result that the weakly self-dual Kähler surfaces are included in a bigger class, that of Kähler surfaces admitting a Hamiltonian 2-form[2]. These connections between surfaces are presented more suggestively in the following diagram:



[1]The fact that any Kähler-Einstein surface is weakly self-dual is a direct consequence of the Corollary 3.1, because the Einstein condition implies that the Ricci form is parallel and the scalar curvature is constant, thus following that the Matsumoto-Tanno equation is satisfied. Because on a Kähler surface the Einstein condition is too "restrictive", in the sense that there are a few Kähler surfaces that are Einstein, it is important to consider generalizations of these surfaces and a possible such generalization is given by the weakly self-dual Kähler surfaces.

[2]According to Proposition 4.2 a Kähler surface is weakly self-dual if and only if the traceless part of the Ricci form is a twistor 2-form, or, equivalently, the Ricci form is a Hamiltonian 2-form.

2) The weakly self-dual Kähler surfaces are a particular case of bi-extremal Kähler surfaces (characterized by the fact that both the scalar curvature and the Pfaffian of the normalized Ricci form are holomorphy potentials), which, at their turn, are a particular case of extremal Kähler surfaces[3] (those whose scalar curvature is a holomorphy potential). These inclusions are presented in the following diagram[4]:



Next we present the main steps of the classification of weakly self-dual Kähler surfaces, that are detailed in the next sections of the paper.[5].

I. We establish equivalent characterizations of weakly self-dual Kähler surfaces, in order to find a characterization which can be easier used to classify these surfaces.

By definition, a Kähler surface is weakly self-dual if and only if its anti-self-dual Weyl tensor is harmonic:

$$\delta W^- = 0.$$

This condition is equivalent to each of the following ones:

- The Cotton-York tensor is self-dual: $C^- = 0$;

[3]This fact together with the local form we find for weakly self-dual Kähler surfaces allows us to give explicit examples of extremal metrics with non-constant scalar curvature.

[4]In this diagram the surfaces that are at the intersection of those having constant scalar curvature and those weakly self-dual have the property that their Ricci tensor is parallel (this follows from the Matsumoto-Tanno equation).

[5]We will use in the sequel notions and notations that are defined and explained in the body of the paper.

- The traceless part of the Ricci form, ρ_0, satisfies the Matsumoto-Tanno equation:

$$\nabla_X \rho_0 = -\frac{1}{2} ds\,(X)\,\omega + \frac{1}{2}\,(ds \wedge JX - Jds \wedge X)\,;$$

- ρ_0 is a twistor 2-form;
- On the set $M_0 := \{x \in M \mid \rho_0(x) \neq 0\}$ the Hermitian structure $(\lambda^{-2}g, I)$ is Kähler, where $\rho_0 = \lambda\omega_I$;
- ρ is a Hamiltonian 2-form.

II. The study of the properties of Kähler surfaces admitting a Hamiltonian 2-form.

Let φ be a Hamiltonian 2-form, *i.e.* φ is J-invariant, closed and φ_0 is a twistor 2-form, to which we associate the normalized 2-form $\tilde{\varphi}$:

$$\varphi = \varphi_0 + \frac{3}{2}\sigma\omega \;\rightsquigarrow\; \tilde{\varphi} = \frac{1}{2}\varphi_0 + \frac{1}{4}\sigma\omega,$$

which has the trace $\mathrm{tr}(\tilde{\varphi}) = \sigma$ and the Pfaffian $\pi := \mathrm{pf}(\tilde{\varphi}) = \frac{\sigma^2}{4} - \lambda^2$, where $\lambda = \frac{|\varphi_0|}{\sqrt{2}}$. The following important properties hold:

(a) The trace σ and the Pfaffian π are Poisson-commuting holomorphy potentials (*cf.* Theorem 4.1).

Thus it yields the existence of two Hamiltonian Killing vector fields:

$$K_1 = J\,\mathrm{grad}\,\sigma, \quad K_2 = J\,\mathrm{grad}\,\pi,$$

which commute: $[K_1, K_2] = 0$ and are orthogonal with respect to the Kähler form ω: $\omega(K_1, K_2) = 0$. In general, these Killing vector fields are not necessarily non-zero or independent. Since $K_1^{1,0}$ and $K_2^{1,0}$ are holomorphic vector fields, we identify on each connected component of the manifold the following 3 possible cases:

(1) $K_1 \wedge K_2$ is non-zero on a dense open set;
(2) $K_1 \wedge K_2$ vanishes identically, but K_1 is non-zero on a dense open set;
(3) K_1 and K_2 vanish identically.

In the third case, the Hamiltonian 2-form φ doesn't contain, in general, much information about the geometry of the manifold (it could be a constant multiple of the Kähler form), but in the first two cases we obtain an explicit classification at the steps IV and VI respectively.

(b) Writing $\sigma = \xi + \eta$ and $\pi = \xi\eta$, then on each connected component of the manifold where φ_0 does not vanish identically, $d\xi$ and $d\eta$ are orthogonal.

III. We particularize the results obtained at the previous step for weakly self-dual Kähler surfaces, whose Ricci form is a Hamiltonian

2-form:

$$\rho = \rho_0 + \frac{3}{2} s\omega \ \rightsquigarrow\ \tilde{\rho} = \frac{1}{2}\rho_0 + \frac{1}{4} s\omega,$$

where s is the normalized scalar curvature: $s = \frac{scal}{6}$.

From property (a) it follows that $s = \mathrm{tr}(\tilde{\rho})$ and $p := \mathrm{pf}(\tilde{\rho})$ are holomorphy potentials, *i.e.* the weakly self-dual Kähler surface is bi-extremal.

Based on the properties already obtained for weakly self-dual Kähler surfaces, we get a first rough classification:

On each connected component of a weakly self-dual Kähler surface (M, g, J, ω) holds one of the following cases:

(1) $\rho_0 = 0$: (g, J) is Kähler -Einstein;
(2) $\rho_0 \neq 0$, $s = const.$: (g, J) is locally the Kähler product of two Riemann surfaces of constant scalar curvatures;
(3) $s \neq const.$ and g is self-dual ($W^- = 0$);
(4) W^- and ρ_0 don't vanish anywhere: the Kähler metric ($\bar{g} = \lambda^{-2} g, I$) given by Proposition 4.1 is extremal and globally defined on M.

IV. The local classification of Kähler surfaces admitting a Hamiltonian 2-form, whose associated Killing vector fields, K_1 and K_2, are linearly independent.

- The local explicit description of an ortho-toric Kähler surface[6] (M, g, J, ω) (*cf.* Proposition 6.2), given by the formulae (6.1)-(6.4), which depend on two arbitrary functions of one variable (F and G):

$$g = (\xi - \eta)\left(\frac{d\xi^2}{F(\xi)} - \frac{d\eta^2}{G(\eta)}\right) + \frac{1}{\xi - \eta}\left(F(\xi)(dt + \eta dz)^2 - G(\eta)(dt + \xi dz)^2\right),$$

$$Jd\xi = \frac{F(\xi)}{\xi - \eta}(dt + \eta dz), \qquad Jdt = -\frac{\xi d\xi}{F(\xi)} - \frac{\eta d\eta}{G(\eta)},$$

$$Jd\eta = \frac{G(\eta)}{\eta - \xi}(dt + \xi dz), \qquad Jdz = \frac{d\xi}{F(\xi)} + \frac{d\eta}{G(\eta)},$$

$$\omega = d\xi \wedge (dt + \eta dz) + d\eta \wedge (dt + \xi dz).$$

- Establishing the following equivalence for Kähler surfaces (*cf.* Theorem 6.1):

  (M, g, J, ω) is ortho-toric $\Longleftrightarrow$ it admits a Hamiltonian 2-form whose associated Killing vector fields are independent,

[6] A Kähler surface is called ortho-toric if it admits two Hamiltonian Killing vector fields whose momentum maps $\xi\eta$ and $\xi + \eta$ Poisson commute and $d\xi \perp d\eta$.

thus showing that the latter are also locally described by the formulae (6.1)-(6.4).

V. Using the results from step IV we obtain the local classification of weakly self-dual Kähler surfaces in the case when the scalar curvature, s, and the Pfaffian of the normalized Ricci form, p, are independent.

We compute the scalar curvature of an ortho-toric Kähler surface using the explicit formula obtained at the previous step and establish the following equivalences for an ortho-toric Kähler surface M:

- M is extremal $\Longleftrightarrow$ F and G have the following form:
$$F(x) = kx^4 + lx^3 + Ax^2 + B_1 x + C_1,$$
$$G(x) = kx^4 + lx^3 + Ax^2 + B_2 x + C_2.$$
- M is weakly self-dual $\Leftrightarrow$ M is bi-extremal $\Leftrightarrow$ F and G as above and $B_1 = B_2$,

thus obtaining the local description of a weakly self-dual Kähler surface with s and p independent.

VI. The local classification of Kähler surfaces admitting a Hamiltonian 2-form, in the case when the associated Killing vector fields, K_1 and K_2, are linearly dependent and K_1 doesn't vanish identically.

- The local description of a Kähler surface of Calabi type[7] (*cf.* Proposition 6.6, given by the formulae (6.40)-(6.41), depending on an arbitrary metric g_Σ on a Riemann surface, on an arbitrary strictly positive function w and on two real constants a, b:
$$g = (az - b)g_\Sigma + w(z)dz^2 + w(z)^{-1}(dt + \alpha)^2,$$
$$\omega = (az - b)\omega_\Sigma + dz \wedge (dt + \alpha).$$
- Establishing the following equivalence for Kähler surfaces (*cf.* Theorem 6.3):
A Kähler surface is of Calabi type if and only if:
(i) it is locally the Kähler product of two Riemannian surfaces, one of which admits a Killing vector field (for $a = 0$); or
(ii) it admits a Hamiltonian 2-form, whose associated Killing vector fields are dependent, but not both zero (for $a = 1$ and $b = 0$).
It thus follows, that a Kähler surface which admits a Hamiltonian 2-form whose associated Killing vector fields are dependent, but not both identically zero, is also locally described by the formulae (6.40)-(6.41), where $a = 1$ and $b = 0$.

[7] A Kähler surface (M, g, J, ω) is of Calabi type if it admits a Hamiltonian Killing vector field which doesn't vanish anywhere, such that the almost Hermitian pair (g, I), where $I = J$ on $\langle K, JK\rangle$ and $I = -J$ on $\langle K, JK\rangle^\perp$, is conformally Kähler.

VII. Using the results obtained at step IV we get the local classification of weakly self-dual Kähler surfaces in the case when s and p are dependent and s is not constant.

We compute the scalar curvature of a Kähler surface of Calabi type, using the explicit local formula obtained for the latter at the previous step and by this formula we establish for s the following equivalences[8] for a Kähler surface of Calabi type (M, g, J, ω), whose Killing vector field is K and which is not locally a Kähler product of Riemannian surfaces:

- s is momentum map for a multiple of the vector field K $\Leftrightarrow$ g_Σ has constant scalar curvature k and V is of the form: $V(z) = A_1 z^4 + A_2 z^3 + kz^2 + A_3 z + A_4$.
- M is weakly self-dual $\Leftrightarrow$ M is bi-extremal $\Leftrightarrow$ g_Σ and V as above and $A_3 = 0$,

obtaining this way the local description of a weakly self-dual Kähler surface with s and p dependent and s is not constant.

Thus it yields the classification of weakly self-dual Kähler surfaces in the three cases, identified according to the Killing vector fields associated to the Hamiltonian 2-form ρ:

(1) K_1, K_2 linearly independent (step V);
(2) K_1, K_2 linearly dependent, K_1 not identically zero (step VII);
(3) K_1 and K_2 vanish identically: in this case the existence of an arbitrary Hamiltonian 2-form, whose associated Killing vector fields are K_1 and K_2, doesn't bring much information about the geometry of the surface, but, for the Ricci form, it yields[9] that the Ricci tensor is parallel, so that (M, g, J, ω) is either Kähler-Einstein, or locally a Kähler product of Riemannian surfaces.

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[8]We keep the same notations as at step VI, so that g_Σ is a metric on a Riemann surface and $V(z) := \frac{z}{w(z)}$ is a strictly positive function, which locally describe the Kähler structure. These equivalences are established in Proposition 6.7.

[9]The vanishing of the vector field $K_1 := J \operatorname{grad} s$ means that s is constant and from the Matsumoto-Tanno equation it follows that the form ρ_0 is parallel, so also the Ricci form, $\rho = \rho_0 + \frac{3}{2} s\omega$, is parallel.

The above described classification of surfaces admitting a Hamiltonian 2-form and the way the formulae giving the local description of the Kähler structures particularize in the case of weakly self-dual Kähler surfaces can be briefly presented as follows:

$K_1 = 0$ $K_2 = 0$	$K_1 \wedge K_2 = 0,\ K_1 \neq 0$	$K_1 \wedge K_2 \neq 0$
	-Kähler surfaces of Calabi type, which are not Kähler products	-Ortho-toric Kähler surfaces
	-The Kähler structure is given by a metric g_Σ on a Riemann surface and a function $V > 0$	-The Kähler structure is given by two arbitrary functions F and G
Ric-parallel	V- polynomial of degree ≤ 4 with the coefficient of the quadartic term equal to k=the constant curvature of the metric g_Σ	F,G-polynomials of degree ≤ 4 with $F - G = const.$

Kähler surfaces admitting a Hamiltonian 2-form

Weakly self-dual Kähler surfaces

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