Special Geometric Structures on Riemannian Manifolds

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Introduction

This habilitation thesis addresses several problems concerning special geometric structures on Riemannian manifolds. The general framework into which this work fits is the question of finding "good" or "special" metrics on smooth manifolds, whereas these terms are considered in a large sense.

The purpose of this introduction is, on the one hand, to briefly give an overview of the broader differential geometric context which made possible and motivated the study of the problems dealt with in this thesis, and on the other hand, to point out the relevance and the interrelation of the obtained results in this more general context. For a more precise presentation of the results we refer to the introduction of each chapter.

There are different possible ways to define what "good" or "special" metrics should be. Very vaguely, one could require for such metrics to exist on "many" (compact) manifolds and at the same time to have distinguished properties which ensure that there are not "too many" of them on a fixed (compact) manifold. A very nice justification why Einstein metrics, defined as having constant Ricci curvature, can be considered as the "best" metrics on compact manifolds is given by A. Besse in the introduction of the book *Einstein manifolds*, [17]. Since there are just a few groups, namely those in the Berger-Simons list, which occur as non-generic irreducible holonomy groups of non-locally-symmetric metrics, the corresponding metrics can be also regarded as being "special". The reduction of the holonomy group can be characterized by the existence of certain parallel geometric structures compatible with the metric, like for instance a parallel complex structure in the Kähler case or a parallel spinor or form for the other holonomy groups. These conditions can be weakened in various ways in order to obtain more flexibility for the existence of such metrics. One possible generalization is to consider G-structures, defined as a reduction only of the structure group to some subgroup of the orthogonal group. For example, when G is the unitary group, we obtain an almost Hermitian structure. Another possibility is to look for the existence of particular spinors of a spin or more generally spin^c structure, satisfying a weaker equation than the one for parallel spinors, like *e.g.* Killing spinors. In a different vein, distinguished metrics are also those having a lot of symmetries, such as toric Kähler metrics.

When studying special geometric structures, an important question is to find topological obstructions for a manifold to admit such structures. Apart from restrictions regarding for example the Betti numbers or the fundamental group, the issue of whether a manifold is formal or not has gained a lot of interest, in particular after the seminal work [27] of P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, where they proved that compact Kähler manifolds are formal.

In the sequel we outline the topics of this work from the above mentioned point of view, namely as possible attempts to detect special metrics or special geometric structures, like homogeneous Clifford structures, Einstein locally conformally Kähler metrics, Kähler-Einstein metrics with special spinors, toric Vaisman metrics or formal metrics.

Clifford structures. The notion of Clifford structure on a Riemannian manifold was introduced by A. Moroianu and U. Semmelmann, [59], and is in a certain sense dual to spin structures. Whereas in spin geometry, the spinor bundle is a representation space of the Clifford algebra bundle of the tangent bundle, in the case of (even) Clifford structures the roles are reversed, it is the tangent bundle of the manifold which becomes a representation space of the (even) algebra bundle of the so-called Clifford bundle (whose rank is called the rank of the Clifford structure). More precisely, a rank r Clifford structure is a rank r subbundle of the skew-symmetric endomorphisms of the tangent bundle, locally spanned by anti-commuting almost complex structures.

As particular cases, we obtain for rank 1, almost Hermitian structures, and for rank 2, almost quaternion-Hermitian structures. Clifford structures are in particular G-structures, namely when there is a reduction of the structure group to $\mathrm{Spin}(r)\cdot C(\mathrm{Pin}(r))$, where $C(\mathrm{Pin}(r))$ denotes the centralizer in the special orthogonal group. The special class of so-called parallel Clifford structures, whose classification was carried out in [59], generalizes hyper-Kähler structures ($r = 2$) and quaternion-Kähler structures ($r = 3$).

The next important subclass is represented by homogeneous Clifford structures. In [57] it is shown that the rank of an even homogeneous Clifford structure on a compact homogeneous manifold of non-vanishing Euler characteristic cannot exceed 16. Moreover, for the ranks $9, 10, 12$ and 16, it is proven that the only manifolds carrying such structures are the so-called Rosenfeld's elliptic projective planes, *cf.* [68]. The other extreme case, namely for rank 3, *i. e.* homogeneous almost quaternion-Hermitian manifolds of non-vanishing Euler characteristic, are classified in [58]. In fact, the only such spaces are: the quaternionic symmetric spaces of Wolf (*cf.* [75]), $\mathbb{S}^2 \times \mathbb{S}^2$ and the complex quadric $\mathrm{SO}(7)/\mathrm{U}(3)$. The methods used in both articles [57] and [58] are of representation-theoretical nature, the main idea behind the classification being that the special configuration of the weights of the spinorial representation is not compatible with the usual integrality conditions of root systems. The remaining cases for intermediary ranks are still open. It would be particularly interesting if a non-symmetric space occurs for one of the left ranks.

Some of the above examples of Clifford structures have been further investigated. F. M. Cabrera and A. Swann, [21], showed that on $\mathrm{SO}(7)/\mathrm{U}(3)$, which is also the twistor space of the six sphere, there is exactly a one-dimensional family of $\mathrm{SO}(7)$-invariant almost quaternion-Hermitian structures (with fixed volume). Moreover, they determined the types of their intrinsic torsion, according to the Gray-Hervella classes introduced in [36]. M. Parton and P. Piccinni [64] studied in more details the projective plane over the complex octonions, namely $\mathrm{E}_6/(\mathrm{Spin}(10)\cdot \mathrm{U}(1))$, for which an explicit description of the Clifford bundle is given. Furthermore, using this description, they constructed a canonical differential 8-form associated with the holonomy $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$, which represents a generator of the cohomology ring.

Spin^c structures. Spin geometry played over the last fifty years a relevant role in both geometry and theoretical physics. Its importance was highlighted by the classical Atiyah-Singer index formula for Dirac operators, which provided a fruitful link between the topology, geometry and analysis of the underlying manifold. Moreover, as mentioned above, most of the geometries with special holonomy also have a spinorial characterization.

The condition for a manifold to be spin, which is equivalent to the vanishing of its second Stiefel-Whitney class, is restrictive for important classes of manifolds. For instance, the complex projective space is spin if and only if its complex dimension is odd. Therefore, one often considers a slight generalization as a complex analogue, namely the so-called spin^c structures, when flexibility is gained by the existence of an auxiliary complex line bundle. The interest in these structures increased starting with the developement of the Seiberg-Witten theory, [69]. Every orientable 4-dimensional manifold admits spin^c-structures, as it was proven by P. Teichner and E. Vogt, [72], and previously by F. Hirzebruch and H. Hopf, [43], in the closed case. Also all almost complex manifolds carry canonical spin^c structures. Special spin^c spinors, generalizing for instance parallel or Killing spinors, play an important role in eigenvalue estimates of the spin^c Dirac operator. M. Herzlich and A. Moroianu, [41], extended the classical estimate of T. Friedrich, [34], to compact Riemannian spin^c manifolds. O. Hijazi, S. Montiel and F. Urbano, [42] constructed spin^c structures on Kähler-Einstein manifolds of positive scalar curvature, which carry so-called Kählerian Killing spin^c spinors. They conjectured that the existence of such spinors is related to a lower bound of the spin^c Dirac operator. In joint work with R. Nakad, [60], we gave a positive answer to this conjecture as an application of refined estimates for the square of the spin^c Dirac operator restricted to the eigenbundles of the Kähler form. These estimates generalize known results for spin structures, like the work of K.-D. Kirchberg, [46], on compact Kähler-Einstein manifolds of even complex dimension or the results in [65] concerning refined estimates on Kähler manifolds for the Dirac operator and the geometric description of the limiting manifolds.

Locally conformally Kähler structures. It is well-known that the existence of a Kähler metric on a closed manifold imposes strong topological restrictions. One natural possibility to relax the Kähler condition and to make it flexible to conformal changes is the following: consider complex manifolds endowed with a Hermitian metric, which is around each point conformal to a Kähler metric. These are called locally conformally Kähler structures (shortly lcK). This is equivalent to the existence of a closed 1-form θ, called the Lee form, which is equal to the differentials of the logarithms of the local conformal factors. While it was established by N. Buchdahl, [20], and A. Lamari, [51], that a complex surface admits a Kähler metric if and only if its first Betti number is even, the conjecture that every complex surface with odd first Betti number carries a compatible lcK structure was disproved by F. Belgun, [14], who showed that some Inoue surfaces do not admit any lcK structure. Moreover, he proved that all Hopf surfaces carry a compatible lcK metric, which was showed before for primary Hopf surfaces by

P. Gauduchon and L. Ornea, [35].

In joint work with F. Madani and A. Moroianu, we considered three classification problems in locally conformally Kähler geometry. Firstly, we gave the complete geometric description of conformal classes on compact manifolds, which contain two non-homothetic Kähler metrics (then necessarily with respect to two non-conjugate complex structures). It turns out that all such manifolds are given by an Ansatz reminiscent of Calabi's construction in [22] on a certain projective line bundle over a Hodge manifold.

Secondly, we classify compact manifolds admitting Einstein (non-Kähler) locally conformally Kähler metrics. One may regard this result as a possible further investigating direction related to the understanding of Kähler-Einstein metrics or Einstein globally conformally Kähler metrics. We considered the left case of locally conformally Kähler manifolds and showed that an Einstein lcK metric on a compact manifold has positive scalar curvature and is necessarily already globally conformal to a Kähler metric.

Compact Kähler Ricci-flat manifolds have been well-understood as a consequence of the famous Calabi Conjecture, whose proof was completed by S.T. Yau, [76]. The case of negative first Chern class was settled by T. Aubin, [10] and S.T. Yau, [76]. If the first Chern class of a compact Kähler manifold is positive, there exist other obstructions to the existence of Kähler-Einstein metrics, as the example of the blow-up of the complex projective space at one point shows. The case of compact complex surfaces was completely classified by G. Tian, [73], who showed that the only ones carrying Kähler-Einstein metrics are the following: the product of two projective lines, the projective plane and its blow-up at k points, for $3 \leq k \leq 8$.

The Einstein metrics which are globally conformal to a Kähler metric, were also investigated. In complex dimension 2, C. LeBrun, [52], and X. Chen, C. LeBrun and B. Weber, [25], showed that the only compact Hermitian surfaces admitting an Einstein globally conformally Kähler (but non-Kähler) metric are either the first Hirzebruch surface endowed with the Page metric, [63], or its blow-up at one or two points. In higher dimensions, A. Derdzinski and G. Maschler, [29], showed that the only compact manifolds carrying a Kähler metric conformal (but not homothetic) to an Einstein metric are obtained by the construction of L. Bérard-Bergery, [16].

Thirdly, we classify compact (non-Kähler) lcK manifolds of complex dimension $n \geq 2$ with non-generic holonomy. In the strictly lcK case, *i.e.* when the metric is not globally conformally Kähler, the only possible non-generic holonomy group is $\mathrm{SO}(2n-1)$ and the manifold is Vaisman, *i.e.* its Lee form is parallel. If the metric is globally conformally Kähler, then there are two possibilities for a non-generic holonomy group: either $\mathrm{U}(n)$, in which case for $n \geq 3$ the manifold is constructed by a Calabi-type Ansatz and for $n = 2$ is ambikähler in the sense of [6], or $\mathrm{SO}(2n-1)$ and the manifold is again obtained by an explicit construction.

Toric structures. Over the last years, toric geometry has experienced a huge developement in several fields with different approaches, like differential geometry, algebraic geometry or mathematical physics, where toric manifolds or varieties are a source of examples and a testing ground for conjectures. For instance, important examples of mirror symmetry and the duality correspondence for Landau Ginzburg models are known for toric varieties.

From the point of view of symplectic geometry, toric manifolds provide examples of extremely symmetric and completely integrable Hamiltonian spaces. A main tool in their study is the so-called momentum map, introduced by J. Souriau, [70], and which formalizes the Noether principle, stating that to every symmetry in a mechanical system corresponds a conserved quantity. The famous theorem of Atiyah and Guillemin-Sternberg, [8] and [39], states that the momentum map of a Hamiltonian toric action on a compact symplectic manifold has connected level sets and its image is the convex hull of the set of moments of the fixed points of the action. In fact, compact symplectic toric manifolds have been classified by T. Delzant, [28], who showed that they are in a one-to-one correspondence to the so-called Delzant polytopes, obtained as the image of the momentum map. Such classification results have been afterwards obtained for other toric manifolds. For instance, it has been extended to non-compact symplectic toric manifolds by Y. Karshon and E. Lerman, [44]. A similar correspondence holds for orbifolds, as shown by E. Lerman and S. Tolman, [54], who proved that compact symplectic toric orbifolds are classified by their moment polytopes, together with a positive integer label attached to each of their facets. The analogue in odd dimension, namely the compact contact toric manifolds, are classified by E. Lerman, [53].

The case when the existence of compatible metrics invariant under the toric action is required is also well understood. Namely, V. Guillemin, [38], showed that for compact toric Kähler manifolds the Delzant polytope determines to a certain extent also the Kähler geometry, by giving an explicit combinatorial formula, in terms of the moment data alone. The Guillemin formula was also derived by D. Calderbank, L. David and P. Gauduchon in [23] via the description of toric symplectic orbifolds as symplectic quotients. Using the so-called action-angle coordinates, M. Abreu studied differential geometric properties of toric Kähler metrics in [1] and gave an effective parametrization of all all toric Kähler metrics on compact symplectic toric orbifolds in [2]. As an application, he describes in [1] examples of extremal Kähler metrics. V. Apostolov, D. Calderbank and P. Gauduchon, [7], classified in all dimensions the special class of the so-called orthotoric Kähler manifolds. The hyperkähler case has been investigated by R. Bielawski and A. Dancer in [18], where they determine a Kähler potential and give an explicit expression for the local form of the toric hyper-Kähler metric. The odd-dimensional counterpart, namely toric Sasaki manifolds, were also studied, for instance by M. Abreu, [3], who gives an approach to toric Kähler-Sasaki geometry via cone action-angle coordinates and symplectic potential.

In [67], we considered toric conformal geometry, more precisely toric locally conformally Kähler geometry, and made a first step towards the understanding of manifolds carrying such structures by investigating the special class of toric Vaisman manifolds. An important notion in this setting is the so-called twisted Hamiltonian action, introduced by I. Vaisman, [74]. There is a close relationship between Vaisman manifolds and both Sasaki and Kähler geometries, through the universal or the so-called minimal covering, on the one hand, and the quotient manifold by the canonical distribution spanned by the Lee and the anti-Lee vector field (in the strongly regular case), on the other hand. In [67] it is showed that these relations still hold in the toric context.

Geometric formality of special geometric structures. The interplay between geometric structures and topological properties of the underlying manifold is one of the most fascinating topics in geometry. This includes natural questions, such as: which topological obstructions are imposed by the existence of some geometric structures and under which circumstances are these sufficient?

One important topological feature of a manifold is the so-called formality, which was introduced by D. Sullivan, [71]. Roughly, a manifold is formal if its real (or rational) homotopy type is a formal consequence of its real (resp. rational) cohomology ring. A consequence of formality is that all Massey products vanish. Simply connected compact manifolds of dimension less than or equal to 6 are formal, *cf.* [32], [56]. The study of formality gained a lot of interest in particular after it was proved in [27] that all compact Kähler manifolds are formal, which amounts to the validity of the $\partial\bar{\partial}$-Lemma. This property provides a way of showing that certain symplectic manifolds do not carry Kähler structures. For instance, in [31] is constructed a 6-dimensional compact symplectic solvmanifold which is not formal and in [33] an 8-dimensional non-formal simply connected compact symplectic manifold. K. Hasegawa, [40], showed that a nilmanifold is formal if and only if it is diffeomorphic to a torus. In particular, this shows that among the nilmanifolds, only the torus admits a Kähler metric, fact which was also proved by C. Benson and C. Gordon, [15]. The feature of being formal or not has been afterwards investigated for other geometric structures. D. Chinea, M. de León and J. Marrero, [26], proved that coKähler manifolds are formal. G. Cavalcanti, [24], extended the formality of compact Kähler manifolds to certain compact generalized Kähler manifolds. However, there are still several interesting classes of manifolds, where not much is known about their formality, like for instance locally conformally Kähler manifolds.

In odd dimensions, the formality of Sasaki manifolds was studied by I. Biswas, M. Fernández, V. Muñoz, A. Tralle, [19], who showed a weaker topological property of all compact Sasaki manifolds, namely that all higher than three Massey products vanish. Based on this, they constructed simply connected K-contact non-Sasaki manifolds, as well as first examples of simply connected compact Sasakian manifolds of dimension greater or equal to 7, which are non-formal. Also 3-Sasaki structures have been investigated from this point of view, for instance in [30] it was showed that a simply connected compact 7-dimensional 3-Sasaki is formal if and only if its second Betti number is less than 2.

A geometric assumption which implies the formality of a compact manifold is the existence of a so-called formal metric, *i.e.* having the property that the wedge product of any two harmonic forms is again harmonic, as defined by D. Kotschick in [47]. A manifold carrying such a metric is called geometrically formal. Apart from the compact symmetric spaces, several examples of geometrically formal, as well as of formal non-geometrically formal, homogeneous manifolds were given by D. Kotschick and S. Terzić, [49], [50] and by M. Amann in [4]. Geometric formality of several special classes of metrics has been investigated, like Kähler metrics by P.-A. Nagy, [61], Sasaki metrics by J.-F. Grosjean and P.-A. Nagy, [37], solvmanifolds by H. Kasuya, [45]. In [62], we determine necessary and sufficient conditions for a Vaisman metric on a compact manifold to be formal, as a first step towards the more general question of understanding the formality of locally conformally Kähler manifolds. We also show that a warping product of two formal metrics is again formal if and only if the warping function is constant. In [66] is given an overview over the (at that time) known results on geometric formality. A new impetus on this subject was given by the more recent works on formal metrics satisfying certain positivity assumptions. C. Bär, [11], classified formal metrics of non-negative sectional curvature on 4-dimensional manifolds up to isometry. D. Kotschick, [48], classified manifolds up to dimension 4 that carry both a formal metric and a (possibly non-formal) metric of non-negative scalar curvature. It was proven by M. Amann and W. Ziller, [5], that a homogeneous geometrically formal metric of positive curvature is either symmetric or a metric on a rational homology sphere.

This habilitation thesis comprises six chapters, each of them coinciding either with a published article or with a preprint on arXiv, up to minor changes, such as enumeration of pages, sections, theorems, etc., as follows:

Chapter 1 A. Moroianu, M. Pilca, *Higher rank homogeneous Clifford Structures*, J. London Math. Soc. **87** (2013), 384–400.

Chapter 2 A. Moroianu, M. Pilca, U. Semmelmann, *Homogeneous almost quaternion-Hermitian manifolds*, Math. Ann. **357** (2013), no. 4, 1205–1216.

Chapter 3 R. Nakad, M. Pilca, *Eigenvalue estimates of the* spin^c *Dirac operator and harmonic forms on Kähler-Einstein manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl., **11** (2015), Paper 054, 15 pages.

Chapter 4 F. Madani, A. Moroianu, M. Pilca, *The holonomy of locally conformally Kähler metrics*, arXiv:1511.09212.

Chapter 5 M. Pilca, *Toric Vaisman manifolds*, arXiv:1512.00876.

Chapter 6 L. Ornea, M. Pilca, *Remarks on the product of harmonic forms*, Pacific J. Math. **250** (2011), 353 – 363.

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Chapter 1

Higher rank homogeneous Clifford structures

Andrei Moroianu and Mihaela Pilca

Abstract. We give an upper bound for the rank r of homogeneous (even) Clifford structures on compact manifolds of non-vanishing Euler characteristic. More precisely, we show that if $r = 2^a \cdot b$ with b odd, then $r \leq 9$ for $a = 0$, $r \leq 10$ for $a = 1$, $r \leq 12$ for $a = 2$ and $r \leq 16$ for $a \geq 3$. Moreover, we describe the four limiting cases and show that there is exactly one solution in each case.

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1.1 Introduction

The notion of (even) Clifford structures on Riemannian manifolds was introduced in [6], motivated by the study of Riemannian manifolds with non-trivial curvature constancy (cf. [3]). They generalize almost Hermitian and quaternionic-Hermitian structures and are in some sense dual to spin structures. More precisely:

Definition 1.1 ([6]) *A rank r* Clifford structure *($r \geq 1$) on a Riemannian manifold (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with an algebra bundle morphism $\varphi : \mathrm{Cl}(E, h) \to \mathrm{End}\,(TM)$ which maps E into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.*

A rank r even Clifford structure *($r \geq 2$) on (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with an algebra bundle morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}\,(TM)$ which maps $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.*

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It is easy to see that every rank r Clifford structure is in particular a rank r even Clifford structure, so the latter notion is more flexible.

In general, there exists no upper bound for the rank of a Clifford structure. In fact, Joyce provided a method (cf. [4]) to construct non-compact manifolds with arbitrarily large Clifford structures. However, in many cases the rank is bounded by above. For instance, the Riemannian manifolds carrying *parallel* (even) Clifford structures (in the sense that (E, h) has a metric connection making the Clifford morphism φ parallel) were classified in [6] and it turns out that the rank of a parallel Clifford structure is bounded by above if the manifold is nonflat: Every parallel Clifford structure has rank $r \leq 7$ and every parallel even Clifford structure has rank $r \leq 16$ (cf. [6, Thm. 2.14 and 2.15]). The list of manifolds with parallel even Clifford structure of rank $r \geq 9$ only contains four entries, the so-called Rosenfeld's elliptic projective planes $\mathbb{O}\mathbb{P}^2$, $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$ and $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$, which are inner symmetric spaces associated to the exceptional simple Lie groups F_4, E_6, E_7 and E_8 (cf. [7]) and have Clifford rank $r = 9$, 10, 12 and 16, respectively.

A natural related question is then to look for *homogeneous* (instead of parallel) even Clifford structures on homogeneous spaces $M = G/H$. We need to make some restrictions on M in order to obtain relevant results. On the first hand, we assume M to be compact (and thus G and H are compact, too). On the other hand, we need to assume that H is not too small. For example, in the degenerate case when H is just the identity of G, the tangent bundle of $M = G$ is trivial, and the unique obstruction for the existence of a rank r (even) Clifford structure is that the dimension of G has to be a multiple of the dimension of the irreducible representation of the Clifford algebra Cl_r or Cl^0_r. At the other extreme, we might look for homogeneous spaces $M = G/H$ with $\mathrm{rk}(H) = \mathrm{rk}(G)$, or, equivalently, $\chi(M) \neq 0$. The main advantage of this assumption is that we can choose a common maximal torus of H and G and identify the root system of H with a subset of the root system of G.

In this setting, the system of roots of G is made up of the system of roots of H and the weights of the (complexified) isotropy representation, which are themselves related to the weights of some spinorial representation if G/H carries a homogeneous even Clifford structure. We then show that the very special configuration of the weights of the spinorial representation Σ_r is not compatible with the usual integrality conditions of root systems, provided that r is large enough.

The main results of this paper are Theorem 1.15, where we obtain upper bounds on r depending on its 2-valuation, and Theorem 1.16, where we study the limiting cases $r = 9$, 10, 12 and 16 and show that they correspond to the symmetric spaces $\mathrm{F}_4/\mathrm{Spin}(9)$, $\mathrm{E}_6/(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4)$, $\mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$ and $\mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.

We believe that our methods could lead to a complete classification of homogeneous Clifford structures of rank $r \geq 3$ on compact manifolds with non-vanishing Euler characteristic, eventually showing that they are all symmetric, thus parallel (cf. [6, Table 2]), but a significantly larger amount of work is needed, especially for lower ranks.

1.2 Preliminaries on Lie algebras and root systems

For the basic theory of root systems we refer to [1] and [8].

Definition 1.2 *A set $\mathcal{R}$ of vectors in a Euclidean space $(V, \langle\,\cdot, \cdot\,\rangle)$ is called a* system of roots *if it satisfies the following conditions:*

R1 *$\mathcal{R}$ is finite,* $\operatorname{span}(\mathcal{R}) = V$, $0 \notin \mathcal{R}$.

R2 *If $\alpha \in \mathcal{R}$, then the only multiplies of α in $\mathcal{R}$ are $\pm\alpha$.*

R3 $\frac{2\langle\,\alpha, \beta\,\rangle}{\langle\,\alpha, \alpha\,\rangle} \in \mathbb{Z}$*, for all $\alpha, \beta \in \mathcal{R}$.*

R4 *$s_\alpha : \mathcal{R} \to \mathcal{R}$, for all $\alpha \in \mathcal{R}$ (s_α is the reflection $s_\alpha : V \to V$, $s_\alpha(v) := v - \frac{2\langle\,\alpha, v\,\rangle}{\langle\,\alpha, \alpha\,\rangle}\alpha$).*

Remark 1.3 (Properties of root systems) *Let $\mathcal{R}$ be a system of roots. If $\alpha, \beta \in \mathcal{R}$ such that $\beta \neq \pm\alpha$ and $\|\beta\|^2 \geq \|\alpha\|^2$, then*

$$\frac{2\langle\,\alpha,\, \beta\,\rangle}{\langle\,\beta,\, \beta\,\rangle} \in \{0, \pm 1\}. \tag{1.1}$$

If $\langle\,\alpha,\, \beta\,\rangle \neq 0$, then the following values are possible:

$$\left(\frac{\|\beta\|^2}{\|\alpha\|^2}, \frac{2\langle\,\alpha,\, \beta\,\rangle}{\langle\,\alpha,\, \alpha\,\rangle}\right) \in \{(1, \pm 1), (2, \pm 2), (3, \pm 3)\}. \tag{1.2}$$

Moreover, in this case, it follows that

$$\beta - \operatorname{sgn}\left(\frac{2\langle\,\alpha,\, \beta\,\rangle}{\langle\,\alpha,\, \alpha\,\rangle}\right) k\alpha \in \mathcal{R}, \qquad \textit{for } k \in \mathbb{Z}, 1 \leq k \leq \left|\frac{2\langle\,\alpha,\, \beta\,\rangle}{\langle\,\alpha,\, \alpha\,\rangle}\right|. \tag{1.3}$$

We shall be interested in special subsets of systems of roots and consider the following notions.

Definition 1.4 *A set $\mathcal{P}$ of vectors in a Euclidean space $(V, \langle\,\cdot, \cdot\,\rangle)$ is called a* subsystem of roots *if it generates V and is contained in a system of roots of $(V, \langle\cdot, \cdot\rangle)$.*

It is clear that any subsystem of roots $\mathcal{P}$ is included into a minimal system of roots (obtained by taking all possible reflections), which we denote by $\overline{\mathcal{P}}$.

Let $\mathcal{P}$ be a subsystem of roots of $(V, \langle\,\cdot, \cdot\,\rangle)$. An *irreducible component* of $\mathcal{P}$ is a minimal non-empty subset $\mathcal{P}' \subset \mathcal{P}$ such that $\mathcal{P}' \perp (\mathcal{P} \setminus \mathcal{P}')$. By rescaling the scalar product $\langle\,\cdot, \cdot\,\rangle$ on the subspaces generated by the irreducible components of V one can always assume that the root of maximal length of each irreducible component of $\mathcal{P}$ has norm equal to 1.

Definition 1.5 *A subsystem of roots $\mathcal{P}$ in a Euclidean space $(V, \langle\,\cdot, \cdot\,\rangle)$ is called* admissible *if $\overline{\mathcal{P}} \setminus \mathcal{P}$ is a system of roots.*

For any $q \in \mathbb{Z}$, $q \geq 1$, let $\mathcal{E}_q$ denote the set of all q-tuples $\varepsilon := (\varepsilon_1, \dots, \varepsilon_q)$ with $\varepsilon_j \in \{\pm 1\}$, $1 \leq j \leq q$. The following result will be used several times in the next section.

Lemma 1.6 *Let $q \in \mathbb{Z}$, $q \geq 1$ and $\{\beta_j\}_{j=\overline{1,q}}$ be a set of linearly independent vectors in a Euclidean space $(V, \langle \cdot, \cdot \rangle)$. If $\mathcal{P} \subset \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ is an admissible subsystem of roots, then any two vectors in $\mathcal{P}$ of different norms must be orthogonal.*

Proof. Assuming the existence of two non-orthogonal vectors of different norms, we construct two roots in $\overline{\mathcal{P}} \setminus \mathcal{P}$ whose difference is a root in $\mathcal{P}$, thus contradicting the assumption on $\mathcal{P}$ to be admissible. More precisely, suppose that $\alpha, \alpha' \in \mathcal{P}$, $\alpha' \neq \alpha$, such that $\langle \alpha, \alpha' \rangle > 0$ (a similar argument works if $\langle \alpha, \alpha' \rangle < 0$) and $\|\alpha'\|^2 > \|\alpha\|^2$. From (1.2), it follows that either $\|\alpha'\|^2 = 2\|\alpha\|^2$ and $\langle \alpha, \alpha' \rangle = \|\alpha\|^2$ or $\|\alpha'\|^2 = 3\|\alpha\|^2$ and $\langle \alpha, \alpha' \rangle = \frac{3}{2}\|\alpha\|^2$. In both cases $\left|\frac{2\langle \alpha', \alpha \rangle}{\langle \alpha, \alpha \rangle}\right| \geq 2$ and (1.3) implies that $\alpha' - \alpha, \alpha' - 2\alpha \in \overline{\mathcal{P}}$.

We first check that $\alpha' - \alpha, \alpha' - 2\alpha \notin \mathcal{P}$. The coefficients of β_j in $\alpha' - \alpha$ and in $\alpha' - 2\alpha$ may take the values $\{0, \pm 2\}$, respectively $\{\pm 1, \pm 3\}$, for all $j = 1, \dots, q$. Since $\{\beta_j\}_{j=\overline{1,q}}$ are linearly independent, it follows that $\alpha' - \alpha \notin \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$. Moreover, $\alpha' - 2\alpha \in \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ if and only if the coefficients of each β_j in α' and α are equal, *i.e.* $\alpha' = \alpha$, which is not possible.

On the other hand, $\langle \alpha' - \alpha, \alpha' - 2\alpha \rangle \in \{\|\alpha\|^2, \frac{1}{2}\|\alpha\|^2\}$ and from (1.3) and the admissibility of $\mathcal{P}$, it follows that $\alpha \in \overline{\mathcal{P}} \setminus \mathcal{P}$, yielding a contradiction and finishing the proof.

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ endowed with an $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and let $\mathcal{R}(\mathfrak{g}) \subset \mathfrak{t}^*$ denote its system of roots. It is well known that $\mathcal{R}(\mathfrak{g})$ satisfies the conditions in Definition 1.2. Conversely, every set of vectors satisfying the conditions in Definition 1.2 is the root system of a unique semi-simple Lie algebra of compact type.

If H is a closed subgroup of G with $\mathrm{rk}(H) = \mathrm{rk}(G)$, then one may assume that its Lie algebra $\mathfrak{h}$ contains $\mathfrak{t}$, so the system of roots $\mathcal{R}(\mathfrak{g})$ is the disjoint union of the root system $\mathcal{R}(\mathfrak{h})$ and the set $\mathcal{W}$ of weights of the complexified isotropy representation of the homogeneous space G/H. This follows from the fact that the isotropy representation is given by the restriction to H of the adjoint representation of $\mathfrak{g}$.

Lemma 1.7 *The set $\mathcal{W} \subset \mathfrak{t}^*$ is an admissible subsystem of roots.*

Proof. Indeed, $\overline{\mathcal{W}} \setminus \mathcal{W} = \overline{\mathcal{W}} \cap \mathcal{R}(\mathfrak{h})$, whence $\overline{\overline{\mathcal{W}} \setminus \mathcal{W}} \subset \overline{\mathcal{W}} \cap \overline{\mathcal{R}(\mathfrak{h})} = \overline{\mathcal{W}} \cap \mathcal{R}(\mathfrak{h}) = \overline{\mathcal{W}} \setminus \mathcal{W}$.

We will now prove a few general results about Lie algebras which will be needed later on.

Lemma 1.8 *Let $\mathfrak{h}_1$ be a Lie subalgebra of a Lie algebra $\mathfrak{h}_2$ of compact type having the same rank. If $\alpha, \beta \in \mathcal{R}(\mathfrak{h}_1)$ such that $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_2)$, then $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_1)$.*

Proof. We first recall a general result about roots. Let $\mathfrak{h}$ be a Lie algebra of compact type and $\mathfrak{t}$ a fixed Cartan subalgebra in $\mathfrak{h}$. For any $\alpha \in \mathfrak{t}^*$, let $(\mathfrak{h})_\alpha$ denote the intersection of the nilspaces of the operators $\mathrm{ad}(A) - \alpha(A)$ acting on $\mathfrak{h}$, with A running over $\mathfrak{t}$. By definition, α is a root of $\mathfrak{h}$ if and only if $(\mathfrak{h})_\alpha \neq \{0\}$. Moreover, the Jacobi identity shows that $[(\mathfrak{h})_\alpha, (\mathfrak{h})_\beta] \subseteq (\mathfrak{h})_{\alpha+\beta}$. It is well known that in this case the space $(\mathfrak{h})_\alpha$ is 1-dimensional. Moreover, by [8, Theorem A, p. 48], there exist generators X_α of $(\mathfrak{h})_\alpha$ such that for any $\alpha, \beta \in \mathcal{R}(\mathfrak{h})$ with $\alpha + \beta \in \mathcal{R}(\mathfrak{h})$, the following relation holds: $[X_\alpha, X_\beta] = \pm(q+1)X_{\alpha+\beta}$, where q is the largest integer k such that $\beta - k\alpha$ is a root. In particular, if $\alpha + \beta \in \mathcal{R}(\mathfrak{h})$, then $[(\mathfrak{h})_\alpha, (\mathfrak{h})_\beta] = (\mathfrak{h})_{\alpha+\beta}$.

Let now $\mathfrak{t}$ be a fixed Cartan subalgebra in both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ (this is possible because $\mathrm{rk}(\mathfrak{h}_1) = \mathrm{rk}(\mathfrak{h}_2)$) and let $\alpha, \beta \in \mathcal{R}(\mathfrak{h}_1) \subseteq \mathcal{R}(\mathfrak{h}_2)$, such that $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_2)$. The above result applied to $\mathfrak{h}_2$ implies: $\{0\} \neq (\mathfrak{h}_2)_{\alpha+\beta} = [(\mathfrak{h}_2)_\alpha, (\mathfrak{h}_2)_\beta] = [(\mathfrak{h}_1)_\alpha, (\mathfrak{h}_1)_\beta] \subseteq (\mathfrak{h}_1)_{\alpha+\beta}$, where we use that $(\mathfrak{h}_1)_\alpha = (\mathfrak{h}_2)_\alpha$ for any $\alpha \in \mathcal{R}(\mathfrak{h}_1) \subseteq \mathcal{R}(\mathfrak{h}_2)$. Thus, $(\mathfrak{h}_1)_{\alpha+\beta} \neq \{0\}$, *i.e.* $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_1)$. □

We will also need the following result, whose proof is straightforward.

Lemma 1.9 *(i) Let $k \geq 2$ and let $\mathfrak{h}$ be a Lie algebra of compact type written as an orthogonal direct sum of k Lie algebras: $\mathfrak{h} = \overset{k}{\underset{i=1}{\oplus}} \mathfrak{h}_i$ with respect to some $\mathrm{ad}_\mathfrak{h}$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$. Then, identifying each Lie algebra $\mathfrak{h}_i$ with its dual using $\langle \cdot, \cdot \rangle$ we have $\mathcal{R}(\mathfrak{h}) = \bigcup_{i=1}^{k} \mathcal{R}(\mathfrak{h}_i)$. In particular, every root of $\mathfrak{h}$ lies in one component $\mathfrak{h}_i$.*

(ii) Let α and β be two roots of $\mathfrak{h}$. If there exists a sequence of roots $\alpha_0 := \alpha, \alpha_1, \ldots, \alpha_n := \beta$ $(n \geq 1)$ such that $\langle \alpha_i, \alpha_{i+1} \rangle \neq 0$ for $0 \leq i \leq n-1$, then α and β belong to the same component $\mathfrak{h}_i$.

1.3 The isotropy representation of homogeneous manifolds with Clifford structure

Let $M = G/H$ be a compact homogeneous space. Denote by $\mathfrak{h}$ and $\mathfrak{g}$ the Lie algebras of H and G and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to some $\mathrm{ad}_\mathfrak{g}$-invariant scalar product on $\mathfrak{g}$. The restriction to $\mathfrak{m}$ of this scalar product defines a homogeneous Riemannian metric g on M. Since from now on we will exclusively consider even Clifford structures, and in order to simplify the terminology, we will no longer use the word "even" and make the following:

Definition 1.10 *A* homogeneous Clifford structure *of rank $r \geq 2$ on a Riemannian homogeneous space $(G/H, g)$ is an orthogonal representation $\rho : H \to \mathrm{SO}(r)$ and an H-equivariant representation $\varphi : \mathfrak{so}(r) \to \mathrm{End}^-(\mathfrak{m})$ extending to an algebra representation of the even real Clifford algebra Cl_r^0 on $\mathfrak{m}$.*

Any homogeneous Clifford structure defines in a tautological way an *even Clifford structure* on (M, g) in the sense of Definition 1.1, by taking E to be the vector bundle associated to the H-principal bundle G over M via the representation ρ:

$$E = G \times_\rho \mathbb{R}^r.$$

In order to describe the isotropy representation of a homogeneous Clifford structure we need to recall some facts about Clifford algebras, for which we refer to [5].

The even real Clifford algebra Cl_r^0 is isomorphic to a matrix algebra $\mathbb{K}(n_r)$ for $r \not\equiv 0 \mod 4$ and to a direct sum $\mathbb{K}(n_r) \oplus \mathbb{K}(n_r)$ when r is multiple of 4. The field $\mathbb{K}$ ($=$ $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) and the dimension n_r depend on r according to a certain 8-periodicity rule. More precisely, $\mathbb{K} = \mathbb{R}$ for $r \equiv 0,\ 1,\ 7 \mod 8$, $\mathbb{K} = \mathbb{C}$ for $r \equiv 2,\ 6 \mod 8$ and $\mathbb{K} = \mathbb{H}$ for $r \equiv 3,\ 4,\ 5 \mod 8$, and if we write $r = 8k + q$, $1 \leq q \leq 8$, then $n_r = 2^{4k}$ for $1 \leq q \leq 4$, $n_r = 2^{4k+1}$ for $q = 5$, $n_r = 2^{4k+2}$ for $q = 6$ and $n_r = 2^{4k+3}$ for $q = 7$ or $q = 8$.

Let Σ_r and $\Sigma_r^\pm$ denote the irreducible representations of Cl_r^0 for $r \not\equiv 0 \mod 4$ and $r \equiv 0 \mod 4$ respectively. From the above, it is clear that Σ_r (or $\Sigma_r^\pm$) have dimension n_r over $\mathbb{K}$.

Lemma 1.11 *Assume that $M = G/H$ carries a rank r homogeneous Clifford structure and let $\iota : H \to \mathrm{Aut}(\mathfrak{m})$ denote the isotropy representation of H.*

(i) If r is not a multiple of 4, we denote by ξ the spin representation of $\mathfrak{so}(r) = \mathfrak{spin}(r)$ on the spin module Σ_r and by $\mu = \xi \circ \rho_*$ its composition with ρ_*. Then the infinitesimal isotropy representation ι_* on $\mathfrak{m}$ is isomorphic to $\mu \otimes_{\mathbb{K}} \lambda$ for some representation λ of $\mathfrak{h}$ over $\mathbb{K}$.

(ii) If r is multiple of 4, we denote by $\xi^\pm$ the half-spin representations of $\mathfrak{so}(r) = \mathfrak{spin}(r)$ on the half-spin modules $\Sigma_r^\pm$ and by $\mu_\pm = \xi^\pm \circ \rho_*$ their compositions with ρ_*. Then the infinitesimal isotropy representation ι_* on $\mathfrak{m}$ is isomorphic to $\mu_+ \otimes_{\mathbb{K}} \lambda_+ \oplus \mu_- \otimes_{\mathbb{K}} \lambda_-$ for some representations $\lambda_\pm$ of $\mathfrak{h}$ over $\mathbb{K}$.

Proof. (i) Consider first the case when r is not a multiple of 4. By definition, the H-equivariant representation $\varphi : \mathfrak{so}(r) \to \mathrm{End}^-(\mathfrak{m})$ extends to an algebra representation of the even Clifford algebra $\mathrm{Cl}_r^0 \simeq \mathbb{K}(n_r)$ on $\mathfrak{m}$. Since every algebra representation of the matrix algebra $\mathbb{K}(n)$ decomposes in a direct sum of irreducible representations, each of them isomorphic to the standard representation on $\mathbb{K}^n$, we deduce that φ is a direct sum of several copies of Σ_r. In other words, $\mathfrak{m}$ is isomorphic to $\Sigma_r \otimes_{\mathbb{K}} \mathbb{K}^p$ for some p, and φ is given by $\varphi(A)(\psi \otimes v) = (\xi(A)\psi) \otimes v$. We now study the isotropy representation ι_* on $\mathfrak{m} = \Sigma_r \otimes_{\mathbb{K}} \mathbb{K}^p$. Note that when $\mathbb{K} = \mathbb{H}$ is non-Abelian, some care is required in order to define the tensor product of representations over $\mathbb{K}$.

The H-equivariance of φ is equivalent to:

$$\iota(h) \circ \varphi(A) \circ \iota(h)^{-1} = \varphi(\rho(h)A), \qquad \forall A \in \mathfrak{so}(r), \forall h \in H.$$

Differentiating this relation at $h = 1$ yields

$$\iota_*(X) \circ \varphi(A) - \varphi(A) \circ \iota_*(X) = \varphi(\rho_*(X)A), \qquad \forall A \in \mathfrak{so}(r), \forall X \in \mathfrak{h}.$$

On the other hand,

$$\varphi(\rho_*(X)A) = \varphi([\rho_*(X), A]) = [\varphi(\rho_*(X)), \varphi(A)] = [\mu(X), \varphi(A)],$$

so

$$[\iota_*(X) - \mu(X), \varphi(A)] = 0, \qquad \forall A \in \mathfrak{so}(r), \forall X \in \mathfrak{h}. \tag{1.4}$$

We denote by $\lambda := \iota_* - \mu$. If $\{v_i\}$ denotes the standard basis of $\mathbb{K}^p$ we introduce the maps $\lambda_{ij} : \mathfrak{h} \to \mathrm{End}_{\mathbb{K}}(\Sigma_r)$ by

$$\lambda(X)(\psi \otimes v_i) = \sum_{j=1}^{p} \lambda_{ji}(X)(\psi) \otimes v_j.$$

The previous relation shows that $\lambda_{ij}(X)$ commutes with the Clifford action $\xi(A)$ on Σ_r for every $A \in \mathfrak{so}(r)$, so it belongs to $\mathbb{K}$. The matrix with entries $\lambda_{ij}(X)$ thus defines a Lie algebra representation $\lambda : \mathfrak{h} \to \mathrm{End}_{\mathbb{K}}(\mathbb{K}^p)$ such that

$$\iota_*(X)(\psi \otimes v) = \mu(X)(\psi) \otimes v + \psi \otimes \lambda(X)(v), \qquad \forall X \in h, \forall \psi \in \Sigma_r, \forall v \in \mathbb{K}^p.$$

This proves the lemma in this case.

(ii) If r is multiple of 4, the even Clifford algebra Cl_r^0 has two inequivalent algebra representations $\Sigma_r^{\pm}$. One can write like before $\mathfrak{m} = \Sigma^+ \otimes_{\mathbb{K}} \mathbb{K}^p \oplus \Sigma^- \otimes_{\mathbb{K}} \mathbb{K}^{p'}$ for some $p, p' \geq 0$, and φ is given by $\varphi(A)(\psi^+ \otimes v + \psi^- \otimes v') = (\xi^+(A)\psi^+) \otimes v + (\xi^-(A)\psi^-) \otimes v'$. The rest of the proof is similar, using the fact that every endomorphism from $\Sigma_r^{\pm}$ to $\Sigma_r^{\mp}$ commuting with the Clifford action of $\mathfrak{so}(r)$ vanishes. $\square$

Let us introduce the ideals $\mathfrak{h}_1 := \ker(\rho_*)$ and $\mathfrak{h}_2 := \ker(\lambda)$ of $\mathfrak{h}$. Since the isotropy representation is faithful, $\mathfrak{h}_1 \cap \mathfrak{h}_2 = 0$ and it is easy to see that $\mathfrak{h}_1$ is orthogonal to $\mathfrak{h}_2$ with respect to the restriction to $\mathfrak{h}$ of any $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product. Denoting by $\mathfrak{h}_0$ the orthogonal complement of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ in $\mathfrak{h}$ we obtain the following orthogonal decomposition:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{1.5}$$

and the corresponding splitting of the Cartan subalgebra of $\mathfrak{h}$: $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2$.

Lemma 1.11 yields further a description of the weights of the isotropy representation of homogeneous spaces with Clifford structure. We assume from now on that $\mathrm{rk}(G) = \mathrm{rk}(H)$ and choose a common Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$. The system of roots of $\mathfrak{g}$ is then the disjoint union of the system of roots of $\mathfrak{h}$ and the weights of the complexified isotropy representation. Since each weight is simple (cf. [8, p. 38]) we deduce that all weights of $\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ are simple.

If r is not multiple of 4, Lemma 1.11 (i) shows that the isotropy representation $\mathfrak{m}$ is isomorphic to $\mu \otimes_{\mathbb{K}} \lambda$ for some representations μ and λ of $\mathfrak{h}$ over $\mathbb{K}$. In order to express $\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ it will be convenient to use the following convention: If ν is a representation over $\mathbb{K}$, we denote by $\nu^{\mathbb{C}}$ the representation over $\mathbb{C}$ given by

$$\begin{cases} \nu^{\mathbb{C}} = \nu \otimes_{\mathbb{R}} \mathbb{C}, & \text{if } \mathbb{K} = \mathbb{R} \\ \nu^{\mathbb{C}} = \nu, & \text{if } \mathbb{K} = \mathbb{C} \\ \nu^{\mathbb{C}} = \nu, & \text{if } \mathbb{K} = \mathbb{H} \end{cases}$$

where in the last row ν is viewed as complex representation by fixing one of the complex structures. Using the fact that if μ and λ are quaternionic representations, then there is a natural isomorphism between $(\mu \otimes_{\mathbb{H}} \lambda)^{\mathbb{C}}$ and $\mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}}$, one can then write

$$\begin{cases} \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}}, & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{H} \\ \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}} \oplus \bar{\mu}^{\mathbb{C}} \otimes_{\mathbb{C}} \bar{\lambda}^{\mathbb{C}}, & \text{if } \mathbb{K} = \mathbb{C} \end{cases} \tag{1.6}$$

If r is multiple of 4, then $\mathfrak{m} = \mu_+ \otimes_{\mathbb{K}} \lambda_+ \oplus \mu_- \otimes_{\mathbb{K}} \lambda_-$ by Lemma 1.11 (ii), and the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{H}$. Consequently,

$$\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu_+^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda_+^{\mathbb{C}} \oplus \mu_-^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda_-^{\mathbb{C}}. \tag{1.7}$$

Let us denote by $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\} \subset \mathfrak{t}^*$ the weights of the representation $\lambda^{\mathbb{C}}$, defined when r is not a multiple of 4. For $r = 2q+1$ $\lambda^{\mathbb{C}}$ is self-dual, so $\mathcal{A} = -\mathcal{A}$. Moreover, $\mathbb{K} = \mathbb{H}$ if $q \equiv 1$ or $2 \mod 4$, so $p = \sharp\mathcal{A}$ is even, whereas for $q \equiv 0$ or $3 \mod 4$ p might be odd, *i.e.* one of the vectors α_i may vanish.

For $r = 2q$ with q even, we denote by $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \dots, \gamma_{p'}\}$ the weights of the representations $\lambda_\pm^{\mathbb{C}}$. Since they are both self-dual we have $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ and we note that $\mathbb{K} = \mathbb{H}$ for $q \equiv 2 \mod 4$, whence p and p' are even in this case.

Recall now that for $r = 2q+1$, the weights of the complex spin representation $\Sigma_r^{\mathbb{C}}$ are

$$\mathfrak{W}(\Sigma_r^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1 \right\},$$

where $\{e_j\}$ is some orthonormal basis of the dual of some Cartan subalgebra of $\mathfrak{so}(2q+1)$. Similarly, if $r = 2q$ with q odd, the weights of the complex spin representation $\Sigma_r^{\mathbb{C}}$ are

$$\mathfrak{W}(\Sigma_r^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = 1 \right\},$$

and for $r = 2q$ with q even, the weights of the complex half-spin representations $(\Sigma_r^{\pm})^{\mathbb{C}}$ are

$$\mathfrak{W}((\Sigma_r^{+})^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = 1 \right\},$$

$$\mathfrak{W}((\Sigma_r^{-})^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = -1 \right\}.$$

We denote by $\beta_j \in \mathfrak{t}^*$ the pull-back through μ_* of the vectors $\frac{1}{2}e_j$, for $j = 1, \dots, q$. Since $\mu = \xi \circ \rho_*$ (and $\mu_\pm = \xi^\pm \circ \rho_*$ for r multiple of 4), the above relations give directly the weights of $\mu^{\mathbb{C}}$ or $\mu_\pm^{\mathbb{C}}$ as linear combinations of the vectors β_j. Taking into account Lemma 1.7, Lemma 1.11, (1.6)-(1.7) and the previous discussion, we obtain the following description of the weights of the isotropy representation of a homogeneous Clifford structure:

Proposition 1.12 *If there exists a homogeneous Clifford structure of rank r on a compact homogeneous space G/H with $\mathrm{rk}(G) = \mathrm{rk}(H)$, then the set $\mathcal{W} := \mathcal{W}(\mathfrak{m})$ of weights of the isotropy representation is an admissible subsystem of roots of $\mathcal{R}(\mathfrak{g})$ and is of one of the following types:*

(I) If $r = 2q + 1$, then there exists $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\} \subset \mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ such that $\mathcal{W} = \mathcal{A} + \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ and $\sharp\mathcal{W} = p \cdot 2^q$. Moreover, if $q \equiv 1$ or $2 \mod 4$ then p is even, so $\alpha_i \neq 0$ for all i.

(II) If $r = 2q$ with q odd, then $\mathcal{W} = \{(\prod_{j=1}^{q} \varepsilon_j)\alpha_i + \sum_{j=1}^{q} \varepsilon_j \beta_j\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_q}$ and $\sharp\mathcal{W} = p \cdot 2^q$.

(III) If $r = 2q$ with $q \equiv 2 \mod 4$, then there exist $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \dots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ such that

$$\mathcal{W} = \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_q} \bigcup \mathcal{G} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

and $\sharp\mathcal{W} = (p + p') \cdot 2^{q-1}$. In this case one of p or p' might vanish, but p and p' are even, so the vectors α_i and γ_i are all non-zero.

(IV) If $r = 2q$ with $q \equiv 0 \mod 4$ (in this case the semi-spinorial representation is real), then there exist $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \dots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ such that

$$\mathcal{W} = \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_q} \bigcup \mathcal{G} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

and $\sharp\mathcal{W} = (p + p') \cdot 2^{q-1}$. In this case one of p or p' might vanish, as well as one of the vectors α_i or γ_i.

In order to describe the homogeneous Clifford structures we shall now obtain by purely algebraic arguments several restrictions on the possible sets of weights of the isotropy representation given by Proposition 1.12.

Proposition 1.13 *Let $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\}$ and $\mathcal{B} := \{\beta_1, \dots, \beta_q\}$ be subsets in a Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with $\beta_j \neq 0, j = \overline{1,q}$. The following restrictions for q hold:*

(I) If $\mathcal{A} = -\mathcal{A}$ and $\mathcal{P}_1 := \mathcal{A} + \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ is a subsystem of roots, then $q \leq 4$. Moreover, if $q = 4$, then $\alpha_i = 0$ for all $1 \leq i \leq p$.

(II) If q is odd and $\mathcal{P}_2 := \{(\prod_{j=1}^{q} \varepsilon_j)\alpha_i + \sum_{j=1}^{q} \varepsilon_j \beta_j\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_q}$ is a subsystem of roots, then $q \leq 7$. Moreover, if $q = 5$ or $q = 7$, then $\alpha_i \neq 0$ for all $1 \leq i \leq p$.

(III)-(IV) If q is even, $\mathcal{A} = -\mathcal{A}$ and

$$\mathcal{P}_3 := \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

or

$$\mathcal{P}_4 := \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

is a subsystem of roots, then $q \leq 8$. Moreover, if $q = 8$, then $\alpha_i = 0$ for all $1 \leq i \leq p$. Thus, if there exists some $\alpha_i \neq 0$, it follows that $q \leq 6$.

Proof. (I) If there exists $i \in \{1, \ldots, p\}$ such that $\alpha_i \neq 0$, let $\beta_0 := \alpha_i$ and $\beta := \sum_{j=0}^{q} \beta_j$. By changing the signs if necessary, we may assume without loss of generality that β has the largest norm among all elements of $\mathcal{P}_1$ and $\|\beta\|^2 = 1$. From (1.1) it follows that

$$\langle \beta, \beta - 2\beta_j \rangle \in \left\{ 0, \pm\frac{1}{2} \right\}, j = 0, \ldots, q.$$

We then have $\frac{1}{2}(q+1) \geq \sum_{j=0}^{q} \langle \beta, \beta - 2\beta_j \rangle = q - 1$, which shows that $q \leq 3$. If $\alpha_i = 0$ for all $i \in \{1, \ldots, p\}$, it follows by the same argument that $q \leq 4$.

(II) If there exists $i \in \{1, \ldots, p\}$ such that $\alpha_i = 0$, then $\{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q} \subset \mathcal{P}_2$ and it follows from (I) that $q \leq 4$. In particular, this shows that if $q \in \{5, 7\}$, then $\alpha_i \neq 0$, for all $i = 1, \ldots, p$.

Otherwise, if $\alpha_i \neq 0$ for all $i = 1, \ldots, p$, then by denoting $\beta_0 := \alpha_1$, we have $\{(\prod_{j=1}^{q} \varepsilon_j)\alpha_1 + \sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q} = \{\sum_{j=0}^{q} \varepsilon_j \beta_j | \prod_{j=0}^{q} \varepsilon_j = 1\}_{\varepsilon \in \mathcal{E}_{q+1}} \subset \mathcal{P}_2$. This subset is of the same type as those considered in (III)–(IV) with $q + 1$ even and with all $\alpha_i = 0$. It then follows from (III)–(IV) that $q + 1 \leq 8$, so $q \leq 7$.

(III)–(IV): If we denote by $\beta'_j := \beta_{2j-1} + \beta_{2j}$, for $j = 1, \ldots, \frac{q}{2}$, then $\mathcal{A} + \{\sum_{j=1}^{q/2} \varepsilon_j \beta'_j\}_{\varepsilon \in \mathcal{E}_{q/2}} \subset \mathcal{P}_3$ is a subsystem of roots. It then follows from (I) that $q \leq 8$ and the equality is attained only if $\alpha_i = 0$, for all $i = 1, \ldots, p$. The same argument holds for $\mathcal{P}_4$ if we choose $\beta'_1 = \beta_1 - \beta_2$ and $\beta'_j := \beta_{2j-1} + \beta_{2j}$, for $j = 2, \ldots, \frac{q}{2}$. □

We now give a more precise description of the subsystems of roots that may occur in the limiting cases of Proposition 1.13. Namely, we determine all the possible scalar products between the roots.

Lemma 1.14 *(a) Let $\mathcal{P} := \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ be a subsystem of roots with $\sharp\mathcal{P} = 2^q$.*

(i) *If $q = 4$, then the Gram matrix of scalar products $(\langle \beta_i, \beta_j \rangle)_{ij}$ is (up to a permutation of the subscripts and sign changes) one of the following:*

$$\frac{1}{4}\mathrm{id}_4 \text{ or } M_0 := \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} \end{pmatrix}. \tag{1.8}$$

Moreover, if $\mathcal{P}$ is admissible, then only the first case can occur, the Gram matrix is $(\langle \beta_i, \beta_j \rangle)_{ij} = \frac{1}{4}\mathrm{id}_4$ and $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(8))$, $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4) \oplus \mathfrak{so}(4))$.

(ii) *If $q = 3$, then the Gram matrix of scalar products $(\langle \beta_i, \beta_j \rangle)_{ij}$ is (up to a permutation of the subscripts and sign changes) one of the following:*

$$M_1 := \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \text{ or } M_2 := \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & \frac{1}{12} \end{pmatrix} \text{ or } M_3 := \begin{pmatrix} \frac{3}{8} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{8} \end{pmatrix}. \tag{1.9}$$

Moreover, if $\mathcal{P}$ is admissible, then only the first two cases can occur. For the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij} = M_1$ the subsystems of roots are $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(6))$ and $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$ and for $(\langle \beta_i, \beta_j \rangle)_{ij} = M_2$, $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{g}_2)$ and $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$.

(b) *Let $\mathcal{P} := \{\sum_{j=1}^{q} \varepsilon_j \beta_j | \prod_{j=1}^{q} \varepsilon_j = 1\}_{\varepsilon \in \mathcal{E}_q}$ be an admissible subsystem of roots with $\sharp\mathcal{P} = 2^q$. If $q = 8$, then the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ is (up to a permutation of the subscripts and sign changes) equal to $\frac{1}{8}\mathrm{id}_8$.*

Proof. (i) As in the proof of Proposition 1.13, we denote by $\beta := \sum_{j=0}^{q} \beta_j$ and, up to sign changes, we may assume that β has the largest norm among all elements of $\mathcal{P}$ and that this norm is equal to 1. We consider the Gram matrix of scalar products $(\langle \beta_i, \beta_j \rangle)_{ij}$. Since $\langle \beta_j, \beta \rangle = \frac{1}{4}$ for all $j = \overline{1, 4}$, the sum of the elements of each of its lines is $\frac{1}{4}$.

Since $\|\beta\|^2 = 1$ is the largest norm of the roots in $\mathcal{P}$, it follows that the square norms of the other roots may take the following values: $\{1, \frac{1}{2}, \frac{1}{3}\}$, so that

$$\|\beta - 2\beta_i\|^2 = 4\|\beta_i\|^2 \Rightarrow \|\beta_i\|^2 \in \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{12}\right\}, \text{ for all } 1 \leq i \leq 4. \tag{1.10}$$

Let $i, j \in \{1, \ldots, 4\}$, $i \neq j$ and assume that $\|\beta_i\|^2 \geq \|\beta_j\|^2$. As $\langle \beta - 2\beta_i, \beta - 2\beta_j \rangle = 4\langle \beta_i, \beta_j \rangle$, it follows by (1.1) that $\langle \beta_i, \beta_j \rangle \in \{0, \pm\frac{1}{2}\|\beta_i\|^2\}$.

The case $\langle \beta_i, \beta_j \rangle = -\frac{1}{2}\|\beta_i\|^2$ cannot occur, because it leads to the following contradiction:

$$0 < \|\beta - 2\beta_i - 2\beta_j\|^2 = 4\|\beta_i + \beta_j\|^2 - 1 = 4\|\beta_j\|^2 - 1 \leq 0.$$

Assume that there exists $i, j \in \{1, \ldots, 4\}$, $i \neq j$, such that $\langle \beta_i, \beta_j \rangle = \frac{1}{2}\|\beta_i\|^2$. It then follows $\|\beta_i + \beta_j\|^2 = 2\|\beta_i\|^2 + \|\beta_j\|^2 \in \{\frac{1}{2}, \frac{3}{8}, \frac{1}{3}\}$, which combined with the restrictions (1.10) yield the following possible values: either $\|\beta_i\|^2 = \|\beta_j\|^2 = \frac{1}{8}$ or $\|\beta_i\|^2 = \frac{1}{8}, \|\beta_j\|^2 = \frac{1}{12}$. In both cases $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{16}$. Hence, all non-diagonal entries are either 0 or $\frac{1}{16}$ and since the sum of the elements of each line is equal to $\frac{1}{4}$, the case $\|\beta_j\|^2 = \frac{1}{12}$ is excluded. It follows that each line of the Gram matrix $(\langle\, \beta_i,\, \beta_j \,\rangle)_{ij}$ either has all non-diagonal entries equal to 0 and the diagonal term is $\frac{1}{4}$ or two of them are $\frac{1}{16}$, one is 0 and the diagonal term is $\frac{1}{8}$. In particular, $\|\beta_i\|^2 \in \{\frac{1}{4}, \frac{1}{8}\}$, for all $1 \le i \le 4$. Up to a permutation of the subscripts we may assume that $\|\beta_i\|^2 \ge \|\beta_j\|^2$ for all $1 \le i < j \le 4$. The following two cases may occur: either $\|\beta_1\|^2 = \|\beta_2\|^2 = \frac{1}{4}$ or $\|\beta_1\|^2 = \frac{1}{4}$ and $\|\beta_2\|^2 = \frac{1}{8}$.

In the first case at most one non-diagonal term may be $\frac{1}{16}$, showing that they must all be equal to 0. This contradicts $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{2}\|\beta_i\|^2 \neq 0$. In the second case, since we have one non-diagonal term equal to $\frac{1}{16}$, we must have another one and then $\|\beta_3\|^2 = \|\beta_4\|^2 = \frac{1}{8}$. Hence, the corresponding Gram matrix is equal to M_0 defined by (1.8).

The subsystem of roots corresponding to the matrix M_0 is not admissible, since the necessary criterion given by Lemma 1.6 is not fulfilled: the set $\{\beta_i\}_{i=\overline{1,4}}$ is linearly independent (since the Gram matrix is invertible) and β, $\beta - 2\beta_2$ are two roots of the subsystem, which have different norms $\|\beta - 2\beta_2\|^2 = \frac{1}{2} = \frac{1}{2}\|\beta\|^2$ and are not orthogonal: $\langle\, \beta,\, \beta - 2\beta_2 \,\rangle = \frac{1}{2}$.

The only case left is when $\{\beta_i\}_{i=\overline{1,4}}$ is an orthonormal system. In this case the new roots obtained by considering all possible reflections are the roots $\{\pm 2\beta_i\}_{i=\overline{1,4}}$, which are of the same norm and orthogonal to each other and thus build the system of roots of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Then the minimal set of roots is $\overline{\mathcal{P}} = \{\pm \sum\limits_{j=1}^{4} \varepsilon_j \beta_j\} \cup \{\pm 2\beta_i\}_{i=\overline{1,4}}$, which is the system of roots of $\mathfrak{so}(8)$, proving (i).

(ii) If $q = 3$, we assume as above that $\beta := \beta_1 + \beta_2 + \beta_3$ is the element of $\mathcal{P}$ of maximal norm and $\|\beta\|^2 = 1$. From (1.1) it follows that $\langle\, \beta,\, \beta - 2\beta_i \,\rangle \in \{0, \pm\frac{1}{2}\}$, for all $i = \overline{1,3}$, yielding that $\langle\, \beta,\, \beta_i \,\rangle \in \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$. Since $1 = \|\beta\|^2 = \sum_{i=1}^{3} \langle\, \beta_i,\, \beta \,\rangle$, the only possible values (up to a permutation of the subscripts) are $\langle\, \beta_1,\, \beta \,\rangle = \frac{1}{2}$ and $\langle\, \beta_2,\, \beta \,\rangle = \langle\, \beta_3,\, \beta \,\rangle = \frac{1}{4}$. Then, from $\|\beta - 2\beta_i\|^2 \in \{1, \frac{1}{2}, \frac{1}{3}\}$, it follows that

$$\|\beta_1\|^2 \in \left\{\frac{1}{2}, \frac{3}{8}, \frac{1}{3}\right\}, \quad \|\beta_2\|^2, \|\beta_3\|^2 \in \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{12}\right\}. \tag{1.11}$$

Since $\langle\, \beta,\, \beta - 2\beta_1 \,\rangle = 0$ and $\langle\, \beta,\, \beta - 2\beta_2 \,\rangle = \langle\, \beta,\, \beta - 2\beta_3 \,\rangle = \frac{1}{2}$, we obtain the following expressions for the scalar products:

$$2\langle\, \beta_2,\, \beta_3 \,\rangle = \|\beta_1\|^2 - \|\beta_2\|^2 - \|\beta_3\|^2, \tag{1.12}$$

$$2\langle\, \beta_1,\, \beta_2 \,\rangle = \frac{1}{2} + \|\beta_3\|^2 - \|\beta_1\|^2 - \|\beta_2\|^2, \tag{1.13}$$

$$2\langle\, \beta_1,\, \beta_3 \,\rangle = \frac{1}{2} + \|\beta_2\|^2 - \|\beta_1\|^2 - \|\beta_3\|^2. \tag{1.14}$$

The other conditions for the scalar products of roots in $\mathcal{P}$ obtained from (1.1) are: $\langle\, \beta - 2\beta_i,\, \beta - 2\beta_j \,\rangle \in \{0, \pm\frac{1}{2}\max(\|\beta - 2\beta_i\|^2, \|\beta - 2\beta_j\|^2)\}$, for all $1 \le i < j \le 3$, which imply

$$\langle \beta_2, \beta_3 \rangle \in \left\{0, \pm\frac{1}{2}\max(\|\beta_2\|^2, \|\beta_3\|^2)\right\}, \tag{1.15}$$

$$\langle \beta_1, \beta_i \rangle \in \left\{\frac{1}{8}, \frac{1}{8} \pm \frac{1}{2}\max(\|\beta_1\|^2 - \frac{1}{4}, \|\beta_i\|^2)\right\}, i = 2, 3. \tag{1.16}$$

We may assume (up to a permutation) that $\|\beta_2\|^2 \geq \|\beta_3\|^2$. By substituting (1.12)–(1.14) in (1.15)–(1.16) we obtain the following conditions: $\|\beta_1\|^2 - \|\beta_2\|^2 - \|\beta_3\|^2 \in \{0, \pm\|\beta_2\|^2\}$ and $\|\beta_2\|^2 - \|\beta_1\|^2 - \|\beta_3\|^2 + \frac{1}{4} \in \{0, \pm\|\beta_3\|^2\}$, which together with the restrictions (1.11) for the norms yield the following possible values:

$$(\|\beta_1\|^2, \|\beta_2\|^2, \|\beta_3\|^2) \in \left\{\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right), \left(\frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right)\right\}.$$

We thus obtain that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ must be equal to one of the three matrices M_i, $i = \overline{1,3}$, defined by (1.9).

By Lemma 1.6, the subsystem of roots corresponding to M_3 is not admissible, since the set $\{\beta_i\}_{i=\overline{1,3}}$ is linearly independent (its Gram matrix is invertible) and there exist two roots β, $\beta - 2\beta_2$ of different norms $\|\beta - 2\beta_2\|^2 = \frac{1}{2} = \frac{1}{2}\|\beta\|^2$, which are not orthogonal: $\langle \beta, \beta - 2\beta_2 \rangle = \frac{1}{2}$.

The subsystem of roots corresponding to the matrix M_1 is admissible and in this case the minimal set of roots containing $\mathcal{P}$ is obtained by adjoining the roots $\pm 2\beta_2$ and $\pm 2\beta_3$, which also have norm equal to 1, showing that $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(6))$ and $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$.

For the Gram matrix M_2, it follows that $\beta_1 = 2\beta_3$ and $\{\beta_1, \beta_2\}$ are linearly independent. The roots in $\mathcal{P}$ have the following norms: $\|\beta\|^2 = \|\beta - 2\beta_2\|^2 = 1$ and $\|\beta - 2\beta_1\|^2 = \|\beta - \beta_1\|^2 = \frac{1}{3}$ and the roots added by all possible reflections in order to obtain the minimal system of roots $\overline{\mathcal{P}}$ are $\{\pm\beta_1, \pm 2\beta_2\}$ with $\|\beta_1\|^2 = \frac{1}{3}$, $\|2\beta_2\|^2 = 1$ and $\langle \beta_1, 2\beta_2 \rangle = 0$. It thus follows that $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$ and $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{g}_2)$ (cf. [2, p. 32]).

(b) If we denote by $\beta'_j := \beta_{2j-1} + \beta_{2j}$, for $j = 1, \ldots, 4$, then $\{\sum_{j=1}^{4} \varepsilon_j \beta'_j\}_{\varepsilon \in \mathcal{E}_4} \subset \mathcal{P}$ is a subsystem of roots. From (i) it follows that the Gram matrix $(\langle \beta'_i, \beta'_j \rangle)_{i,j}$ is (up to rescaling, reordering and sign change of the vectors β'_j) either $\frac{1}{4}\mathrm{id}$ or the matrix M_0 defined by (1.8).

We claim that the second case cannot occur. Indeed, if this were the case, then $\|\beta_1 + \beta_2\|^2 = \frac{1}{4}$, $\|\beta_{2j-1} + \beta_{2j}\|^2 = \frac{1}{8}$ for $j = 2, 3, 4$, and

$$\langle \beta_3 + \beta_4, \beta_5 + \beta_6 \rangle = \frac{1}{16}. \tag{1.17}$$

For every j and k with $3 \leq j < k \leq 8$, there exists $l \in \{2, 3, 4\}$ such that $\{2l - 1, 2l\} \cap \{j, k\} = \emptyset$. Let $\{s, t\}$ denote the complement of $\{1, 2, j, k, 2l - 1, 2l\}$ in $\{1, \ldots, 8\}$. The Gram matrix of $\{\beta_1 + \beta_2, \beta_{2l-1} + \beta_{2l}, \beta_j - \beta_k, \beta_s - \beta_t\}$ has at least two different values on the diagonal. Again from (i), it follows that the remaining diagonal terms $\|\beta_j - \beta_k\|^2$

and $\|\beta_s - \beta_t\|^2$ must both be equal to $\frac{1}{8}$. Thus, $\|\beta_j - \beta_k\|^2 = \frac{1}{8}$ for all $3 \leq j < k \leq 8$. By the same argument we also obtain $\|\beta_j + \beta_k\|^2 = \frac{1}{8}$ for all $3 \leq j < k \leq 8$. Thus, $\langle \beta_j, \beta_k \rangle = 0$ for all $3 \leq j < k \leq 8$, contradicting (1.17).

This shows that the vectors β'_j are mutually orthogonal. Applying this to different partitions of the set $\{1, \ldots, 8\}$ into four pairs we get that $\beta_j + \beta_k$ is orthogonal to $\beta_s + \beta_t$ for all mutually distinct subscripts j, k, s and t. This clearly implies that $\langle \beta_i, \beta_j \rangle = 0$ for all $i \neq j$. It then also follows that $\|\beta_j\|^2 = \frac{1}{8}$, for $j = 1, \ldots, 8$, proving (b). □

1.4 Homogeneous Clifford Structures of high rank

A direct consequence of Propositions 1.12 and 1.13 is the following upper bound for the rank of a homogeneous Clifford structure:

Theorem 1.15 *The rank r of any even homogeneous Clifford structure on a homogeneous compact manifold G/H of non-vanishing Euler characteristic is less or equal to* 16. *More precisely, the following restrictions hold for the rank depending on its* 2*-valuation:*

(I) If r is odd, then $r \in \{3, 5, 7, 9\}$.

(II) If $r \equiv 2 \mod 4$, then $r \in \{2, 6, 10\}$.

(III) If $r \equiv 4 \mod 8$, then $r \in \{4, 12\}$.

(IV) If $r \equiv 0 \mod 8$, then $r \in \{8, 16\}$.

Proof. We only need to show that in case (II), the rank r is strictly less than 14. This will be done in the proof of Theorem 1.16 (II). □

We further describe the manifolds which occur in the limiting cases for the upper bounds in Theorem 1.15.

Theorem 1.16 *The maximal rank r of an even homogeneous Clifford structure for each of the types (I)-(IV) and the corresponding compact homogeneous manifolds $M = G/H$ (with $\mathrm{rk}(G) = \mathrm{rk}(H)$) carrying such a structure are the following:*

(I) $r = 9$ and M is the Cayley projective space $\mathbb{OP}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$.

(II) $r = 10$ and $M = (\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4)$.

(III) $r = 12$ and $M = (\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$.

(IV) $r = 16$ and $M = (\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.

Proof. Let $M = G/H$ ($\mathrm{rk}(G) = \mathrm{rk}(H)$) carry a homogeneous Clifford structure of rank r.

(I) If r is odd, $r = 2q + 1$, then it follows from Theorem 1.15 (I) that $r \leq 9$.

By Proposition 1.12 (I) and Lemma 1.14 (i), the set $\mathcal{W} := \mathcal{W}(\mathfrak{m})$ of weights of the isotropy representation is $\mathcal{W} = \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ with $\sharp\mathcal{W} = 2^q$ and $\mathcal{R}(\mathfrak{so}(8)) \subseteq \mathcal{R}(\mathfrak{g})$. In particular the representation λ is trivial, so $\mathfrak{h} = \mathfrak{h}_2$ and $\sharp\mathcal{R}(\mathfrak{g}) \geq 24$.

Since $\rho_* : \mathfrak{h}_2 \to \mathfrak{so}(9)$ is injective and $\mathfrak{h} = \mathfrak{h}_2$, it follows that $\mathrm{rk}(\mathfrak{h}) \leq 4$.

If $\mathrm{rk}(\mathfrak{h}) \leq 3$, then $\sharp\mathcal{R}(\mathfrak{g}) \leq 18$ by a direct check in the list of Lie algebras of rank 3. This contradicts the fact that $\sharp\mathcal{R}(\mathfrak{g}) \geq 24$.

Thus, $\mathrm{rk}(\mathfrak{h}) = 4$ and ρ_* is a bijection when restricted to a Cartan subalgebra of $\mathfrak{h}$. On the other hand, from Lemma 1.14 (i), it also follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ is equal to $\frac{1}{4}\mathrm{id}_4$. The new roots obtained by reflections given by (1.3) are the following: $\pm 2\beta_i = \pm e_i \circ \rho_* \in \mathcal{R}(\mathfrak{h})$, for all $i = \overline{1,4}$. As $\mathrm{rk}(\mathfrak{h}) = 4 = \mathrm{rk}(\mathfrak{so}(9))$, we may apply Lemma 1.8 for $\mathfrak{h} \subseteq \mathfrak{so}(9)$ and get that $\{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq 4\}$, which are roots of $\mathfrak{so}(9)$, are also roots of $\mathfrak{h}$. Thus, $\mathfrak{h} = \mathfrak{so}(9)$. The Lie algebra $\mathfrak{g}$, whose system of roots is obtained by joining the system of roots of $\mathfrak{so}(9)$ with the weights of the spinorial representation of $\mathfrak{spin}(9)$, is then exactly $\mathfrak{f}_4$ (cf. [2, p. 55]). We note that we cannot extend $\mathfrak{f}_4$, since there is no other larger Lie algebra of the same rank. Using the fact that the closed subgroup of F_4 corresponding to the above embedding of $\mathfrak{so}(9)$ in $\mathfrak{f}_4$ is $\mathrm{Spin}(9)$ (cf. [2, Thm. 6.1]), we deduce that the only homogeneous manifold carrying a homogeneous Clifford structure of rank 9 is the Cayley projective space $\mathbb{OP}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$.

(II) By Theorem 1.15 (II), $r \leq 14$. We first show that there exists no homogeneous Clifford structure of rank $r = 14$.

Let $r = 2q = 14$. In this case, by Proposition 1.12 (II), the set of weights of the isotropy representation is $\mathcal{W} := \mathcal{W}(\mathfrak{m}) = \left\{ (\prod_{j=1}^{7} \varepsilon_j)\alpha_i + \sum_{j=1}^{7} \varepsilon_j \beta_j \right\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_7}$ with $\sharp\mathcal{W} = p \cdot 2^7$. Proposition 1.13 (II) yields that $\alpha_i \neq 0$, for all $1 \leq i \leq p$.

We claim that the following inclusion holds: $\mathcal{R}(\mathfrak{so}(16)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$. This can be seen as follows. Denoting by $\beta_0 := \alpha_1$ and $\beta := \beta_0 + \cdots + \beta_7$, the set $\mathcal{W}$ contains the following subsystem of roots $\{\sum_{j=0}^{7} \varepsilon_j \beta_j | \prod_{j=0}^{7} \varepsilon_j = 1\}_{\varepsilon \in \mathcal{E}_8}$. From Lemma 1.14 (b) it follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{i,j=\overline{0,7}}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Then, for all $0 \leq i < j \leq 7$ we have $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \frac{1}{2}$, implying by (1.3) that there are new roots $\pm 2(\beta_i + \beta_j) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. Similarly, for any $0 \leq k \leq 7$ distinct from i and j, $\langle \beta - 2\beta_i - 2\beta_k, \beta - 2\beta_j - 2\beta_k \rangle = \frac{1}{2}$ yields the new roots $\pm 2(\beta_i - \beta_j) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. It thus follows that $\mathcal{R}(\mathfrak{so}(16)) = \{\pm 2(\beta_i \pm \beta_j) \,|\, 0 \leq i \leq 7\} \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$.

Since $\mathcal{R}(\mathfrak{so}(16)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W} \subseteq \mathcal{R}(\mathfrak{h})$, it follows that $\mathfrak{so}(16)$ is a Lie subalgebra of $\mathfrak{h}$. Recall the splitting (1.5): $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(14)$. As $\mathfrak{so}(16)$ is a simple Lie algebra, it follows that $\mathfrak{so}(16) \subseteq \mathfrak{h}_1$. In particular, this implies $p \geq 8$.

On the other hand, we show that $p = 1$. Assume that $p \geq 2$. By Lemma 1.14 (b), the Gram matrix of both subsystems of roots $\{\alpha_1, \beta_1, \ldots, \beta_7\}$ and $\{\alpha_2, \beta_1, \ldots, \beta_7\}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Denoting by $a := \langle \alpha_1, \alpha_2 \rangle$, we obtain the following values for the scalar products between the roots containing α_1 and α_2: $\{a + \frac{7}{8}, a + \frac{3}{8}, a - \frac{1}{8}\}$. From (1.1), we know that these values must belong to $\{0, \pm\frac{1}{2}\}$. It then follows that the only possible value for $a = \langle\, \alpha_1,\, \alpha_2 \,\rangle$ is $-\frac{3}{8}$. This leads to a contradiction by computing the following norm: $\|\alpha_1 + \alpha_2\|^2 = -\frac{1}{2} < 0$.

Thus, the case $r = 14$ is not possible.

Now, for rank $r = 10 = 2q$, by Proposition 1.12 (II), the set of weights of the isotropy representation is $\mathcal{W} := \mathcal{W}(\mathfrak{m}) = \{(\prod_{j=1}^{5} \varepsilon_j)\alpha_i + \sum_{j=1}^{5} \varepsilon_j \beta_j\}_{i=\overline{1,p},\varepsilon\in\mathcal{E}_5}$ with $\sharp\mathcal{W} = p \cdot 2^5$. From Proposition 1.13 (II), it then follows that $\alpha_i \neq 0$, for all $1 \le i \le p$. We will further show that $p = 1$ and the following inclusions hold: $\mathcal{R}(\mathfrak{spin}(10) \oplus \mathfrak{u}(1)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$, $\mathcal{R}(\mathfrak{e}_6) \subseteq \overline{\mathcal{W}}$.

Let $\beta_0 := \alpha_1$ and $\beta'_j := \beta_{2j} + \beta_{2j+1}$, for $j = 0, 1, 2$. Then $\{\sum_{j=0}^{2} \varepsilon_j \beta'_j\} \subset \mathcal{W}$ is an admissible subsystem of roots. By Lemma 1.14 (ii), the Gram matrix $B' := (\langle\, \beta'_i,\, \beta'_j \,\rangle)_{i,j=\overline{0,2}}$ is one of the three matrices in (1.9). We may assume that β'_0 is of maximal norm and equal to 1, so that the possible Gram matrices B' are normalized as follows:

$$M_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, M_2 := \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}, M_3 := \begin{pmatrix} 1 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}. \tag{1.18}$$

We first note that the norm of $\beta := \sum_{j=0}^{5} \beta_j$, which is equal to the sum of all elements of the matrix B', may take the following values: $\|\beta\|^2 = 2$ for M_1, $\|\beta\|^2 = 3$ for M_2, $\|\beta\|^2 = \frac{8}{3}$ for M_3. We will show that the last two cases cannot occur.

Let us first assume that $B' = M_2$. In this case $\beta'_0 = 2\beta'_2$, *i.e.* $\beta_0 + \beta_1 = 2(\beta_4 + \beta_5)$. Considering now another pairing by permuting the subscripts $2, 3, 4, 5$, we get a Gram matrix which must also be equal to M_2, since $\|\beta\|^2$ does not change. We may thus assume that $\beta_0 + \beta_1 = 2(\beta_2 + \beta_4)$ and is furthermore equal either to $2(\beta_2 + \beta_5)$ or to $2(\beta_3 + \beta_4)$. In both cases it follows that there exists $i \neq j \in \{2, 3, 4, 5\}$, such that $\beta_i = \beta_j$. Then, for any $k \neq i, j$, the roots $\beta - 2\beta_i - 2\beta_k, \beta - 2\beta_j - 2\beta_k \in \mathcal{W}$ are equal, which contradicts $\sharp\mathcal{W} = p \cdot 2^5$. Thus, $B' \neq M_2$.

Let us now assume that the Gram matrix B' is equal to M_3. Then $\|\beta\|^2 = \frac{8}{3}$ and since this is the maximal norm, it follows that for any other possible pairing of the vectors β_j, the corresponding Gram matrix is either M_1 or M_3 (because the sum of all elements of M_2 is $3 > \frac{8}{3}$).

We consider as above other pairings by permuting the subscripts $\{2, 3, 4, 5\}$. Again by Lemma 1.14 (ii), it follows that the corresponding Gram matrix is one of the matrices in (1.18) and, since $\|\beta\|^2$ does not change, it must also be equal to M_3. In particular, we have:

$$\|\beta_i + \beta_j\|^2 = \frac{1}{3}, \tag{1.19}$$

$$\langle\, \beta_i + \beta_j,\, \beta_k + \beta_l \,\rangle = \frac{1}{6}, \tag{1.20}$$

for any permutation (i, j, k, l) of $(2, 3, 4, 5)$.

Consider now the following pairings of the vectors β_j:

$$\beta''_0 := \beta_0 + \beta_1, \beta''_1 := \pm(\beta_i - \beta_j), \beta''_2 := \pm(\beta_k - \beta_l),$$

where (i,j,k,l) is any permutation of $(2,3,4,5)$ and in each case the signs for β''_1 and β''_2 are chosen such that

$$\|\beta''_0 + \beta''_1 + \beta''_2\|^2 = \max\{\|\beta''_0 \pm \beta''_1 \pm \beta''_2\|^2\}.$$

Then $\{\sum_{j=0}^{2} \varepsilon_j \beta''_j\}$ is a subsystem of roots of $\mathcal{W}$ and by the same argument as above its Gram matrix $B'' := (\langle \beta''_i, \beta''_j \rangle)_{i,j=\overline{0,2}}$ is either M_1 or M_3.

Since $\|\beta''_0\|^2 = 1$, it follows that in both cases the norms of the other two vectors are equal: $\|\beta''_1\|^2 = \|\beta''_2\|^2$, *i.e.* $\|\beta_i - \beta_j\|^2 = \|\beta_k - \beta_l\|^2 \in \{\frac{1}{2}, \frac{1}{3}\}$. By (1.19), we then obtain for any $i,j \in \{2,3,4,5\}$ that $\langle \beta_i, \beta_j \rangle = \frac{1}{4}(\|\beta_i + \beta_j\|^2 - \|\beta_i - \beta_j\|^2) \in \{0, -\frac{1}{24}\}$. It then follows that $\langle \beta_i + \beta_j, \beta_k + \beta_l \rangle < 0$, for any permutation (i,j,k,l) of $(2,3,4,5)$, which contradicts (1.20). Thus, $B' \neq M_3$.

The only possibility left is $B' = M_1$. Then $\|\beta\|^2 = 2$ and since it is the element of maximal norm, it follows that for any other pairing of the vectors β_j, the corresponding Gram matrix is also equal to M_1. We then have $B' = B'' = M_1$, which implies:

$$\langle \beta_0 + \beta_1, \beta_i \pm \beta_j \rangle = 0, \tag{1.21}$$

$$\|\beta_i + \beta_j\|^2 = \|\beta_i - \beta_j\|^2 = \frac{1}{2}, \tag{1.22}$$

$$\langle \beta_i + \beta_j, \beta_k + \beta_l \rangle = \langle \beta_i - \beta_j, \beta_k - \beta_l \rangle = 0, \tag{1.23}$$

for any permutation (i,j,k,l) of $(2,3,4,5)$. From (1.21) it follows that $\langle \beta_0 + \beta_1, \beta_i \rangle = 0$ for all $2 \leq i \leq 5$. From (1.23) it follows that $\langle \beta_i, \beta_j \rangle = 0$, for $2 \leq i < j \leq 5$ and then from (1.22) we obtain $\|\beta_i\|^2 = \frac{1}{4}$, for $2 \leq i \leq 5$.

For pairings of the following form $\beta_0 - \beta_1$, $\beta_i - \beta_j$, $\beta_k + \beta_l$, where again (i,j,k,l) is a permutation of $(2,3,4,5)$, the corresponding Gram matrix must also be equal to M_1. Since $\|\beta_i \pm \beta_j\|^2 = \frac{1}{4}$, for all $2 \leq i < j \leq 5$, it follows that $\|\beta_0 - \beta_1\|^2 = 1$, so $\langle \beta_0, \beta_1 \rangle = 0$. By the above argument applied to this pairing, it follows a similar relation, namely: $\langle \beta_0 - \beta_1, \beta_i \pm \beta_j \rangle = 0$, which together with (1.21) yields $\langle \beta_0, \beta_i \rangle = \langle \beta_1, \beta_j \rangle = 0$, for $2 \leq i \leq 5$. Thus, the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{0 \leq i,j \leq 5}$ is diagonal with $\|\beta_0\|^2 + \|\beta_1\|^2 = 1$ and $\|\beta_i\|^2 = \frac{1}{4}$, for $2 \leq i \leq 5$.

The second element in the decreasing order of the set $\{\|\beta_i + \beta_j\|^2 \,|\, 0 \leq i < j \leq 5\}$ must be, up to a permutation of 0 and 1, of the form $\beta_0 + \beta_k$, for some $2 \leq k \leq 5$. By taking now, for instance, a pairing with first element equal to $\beta_0 + \beta_k$, it follows that its Gram matrix is equal to M_1 and by the same argument as above $\|\beta_1\|^2 = \frac{1}{4}$ and, consequently, $\|\beta_0\|^2 = \frac{3}{4}$. Since we allowed permutations, we have to consider two cases: $\|\alpha_1\|^2 \in \{\frac{3}{4}, \frac{1}{4}\}$.

If $\|\alpha_1\|^2 = \frac{1}{4}$, we may assume that $\|\beta_1\|^2 = \frac{3}{4}$ and $\|\beta_i\|^2 = \frac{1}{4}$, for all $2 \leq i \leq 5$. Since the Gram matrix is now completely known, we compute the scalar products between the roots in $\mathcal{W}$ and by (1.3) we obtain: $\{\pm 2(\alpha_1 \pm \beta_i), \pm 2(\beta_j \pm \beta_i) | \, 2 \leq i < j \leq 5\} \subseteq \overline{\mathcal{W}} \backslash \mathcal{W} \subseteq \mathcal{R}(\mathfrak{h})$. Considering the orthogonal decomposition (1.5) of $\mathfrak{h}$, it follows by Lemma 1.9 (ii) that there is a $k \in \{0,1,2\}$ such that $\{\pm 2(\alpha_1 \pm \beta_i), \pm 2(\beta_j \pm \beta_i) | \, 2 \leq i < j \leq 5\} \subseteq \mathcal{R}(\mathfrak{h}_k)$. Since $\alpha_1 \in \mathfrak{h}_0 \oplus \mathfrak{h}_1$ and $\beta_i \in \mathfrak{h}_0 \oplus \mathfrak{h}_2$, for $1 \leq i \leq 5$, the only possible value is $k = 0$. This implies that $\mathfrak{so}(10) \subseteq \mathfrak{h}_0$ and thus $p \geq 5$. We show that this is not possible.

Assuming that $p \geq 2$ and computing the scalar products between $\alpha_1 + \beta_1 + \cdots + \beta_5$ and $\alpha_2 + \beta_1 \pm (\beta_2 + \beta_3) \pm (\beta_4 + \beta_5)$, we get the following values $\{a + \frac{7}{4}, a + \frac{3}{4}, a - \frac{1}{4}\}$, where $a := \langle\, \alpha_1, \alpha_2 \,\rangle$. By (1.1), we know that $\{a + \frac{7}{4}, a + \frac{3}{4}, a - \frac{1}{4}\} \subseteq \{0, \pm 1\}$. Hence, $a = -\frac{3}{4}$, which implies that $\|\alpha_1 + \alpha_2\|^2 = -1$. Thus, the case $\|\alpha_1\|^2 = \frac{1}{4}$ may not occur.

We then have $\|\alpha_1\|^2 = \frac{3}{4}$ and $\|\beta_i\|^2 = \frac{1}{4}$, for $1 \leq i \leq 5$. Again by computing all possible scalar products, we produce by (1.3) the new roots $\{\pm 2(\beta_i \pm \beta_j) | \, 1 \leq i < j \leq 5\} \subseteq \mathcal{R}(\mathfrak{h})$. By Lemma 1.9 (i) and (ii), there exists a $k \in \{0, 1, 2\}$ such that $\{\pm 2(\beta_i \pm \beta_j) | \, 1 \leq i < j \leq 5\}$ are all roots of one of the components $\mathfrak{h}_k$ of the orthogonal splitting $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ given by (1.5). As $\beta_i \in \mathfrak{h}_0 \oplus \mathfrak{h}_2$ for $1 \leq i \leq 5$, it follows that $k \in \{0, 2\}$. Thus, $\mathcal{R}(\mathfrak{h}_k)$ contains the whole system of roots of $\mathfrak{so}(10)$. On the other hand, $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(10)$. Hence, there are two possibilities: either $\mathfrak{h}_0 = \mathfrak{so}(10)$ and $\mathfrak{h}_2 = 0$ or $\mathfrak{h}_0 = 0$ and $\mathfrak{h}_2 = \mathfrak{so}(10)$.

Let us first note that if $p \geq 2$, then $\langle\, \alpha_i, \alpha_j \,\rangle = -1$, for all $1 \leq i < j \leq p$. By computing the scalar products between the different roots containing α_i, respectively α_j, we obtain the following values: $a := \langle\, \alpha_i, \alpha_j \,\rangle \in \{a + \frac{5}{4}, a + \frac{1}{4}, a - \frac{3}{4}\}$, which by (1.1) must be contained into $\{0, \pm 1\}$. Hence, the only possible value is $a = -\frac{1}{4}$.

In the first case, $\mathfrak{h}_0 = \mathfrak{so}(10)$ implies that $p \geq 5$, which by the above remark leads to the following contradiction: $\|\alpha_1 + \cdots + \alpha_5\|^2 = -\frac{5}{4} < 0$.

Thus, the second case $\mathfrak{h}_2 = \mathfrak{so}(10)$ and $\mathfrak{h}_0 = 0$ must hold. We show that $p = 1$. Assuming $p \geq 2$, we compute $\langle\, \alpha_1 + \beta_1 + \cdots + \beta_5, \alpha_2 + \beta_1 - \beta_2 - \cdots - \beta_5 \,\rangle = -1$, which by (1.3) yields the new mixed root $\alpha_1 + \alpha_2 + 2\beta_1 \in \mathcal{R}(\mathfrak{h})$, contradicting Lemma 1.9 (i) (since $\alpha_1 + \alpha_2 \in \mathfrak{h}_1$ and $\beta_1 \in \mathfrak{h}_2$). Thus, $p = 1$ and $\mathfrak{h}_1 = \mathfrak{u}(1)$.

Concluding, it follows that $\mathfrak{h} = \mathfrak{so}(10) \oplus \mathfrak{u}(1)$. Therefore, $\mathcal{R}(\mathfrak{g}) = \mathcal{W} \cup \mathcal{R}(\mathfrak{so}(10) \oplus \mathfrak{u}(1))$, is exactly the system of roots of $\mathfrak{e}_6$ (cf. [2, p. 57]), hence $\mathfrak{g} = \mathfrak{e}_6$. From [2, Thm. 6.1], the Lie subgroup of E_6 corresponding to the above embedding of $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ in $\mathfrak{e}_6$ is $\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4$, showing that the only homogeneous manifold carrying a homogeneous Clifford structure of rank $r = 10$ is the exceptional symmetric space $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4)$.

(III) By Theorem 1.15 (III), the maximal rank in this case is $r = 12$. For $r = 12 = 2q$, from Proposition 1.12 (III), there exist $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \ldots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ such that the set of weights of the isotropy representation is given by:

$$\mathcal{W}(\mathfrak{m}) = \mathcal{A} + \left\{ \sum_{j=1}^{6} \varepsilon_j \beta_j | \prod_{j=1}^{6} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_6} \bigcup \mathcal{G} + \left\{ \sum_{j=1}^{6} \varepsilon_j \beta_j | \prod_{j=1}^{6} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_6}$$

with $\sharp\mathcal{W} = (p + p') \cdot 2^5$, where one of p or p' might vanish, but the vectors α_i and γ_i are all non-zero.

Assume $p \neq 0$ (otherwise the same argument applies for $p' \neq 0$ by changing the sign of β_1) and denote by

$$\mathcal{W} := \mathcal{A} + \left\{ \sum_{j=1}^{6} \varepsilon_j \beta_j | \prod_{j=1}^{6} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_6},$$

which is an admissible subsystem of roots as in Proposition 1.13 (III), with $\sharp\mathcal{W} = p \cdot 2^5$.

We claim that the following inclusions hold: $\mathcal{R}(\mathfrak{so}(12) \oplus \mathfrak{su}(2)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$, $\mathcal{R}(\mathfrak{e}_7) \subseteq \overline{\mathcal{W}}$.

Let us consider all the subsystems of roots of the following form $\{\sum_{j=1}^{4} \varepsilon_j \beta'_j\} \subset \mathcal{W}$, where $\beta'_1 = \alpha_1$, $\beta'_2 = \beta_1 \pm \beta_2$, $\beta'_3 = \beta_3 \pm \beta_4$, $\beta'_4 = \beta_5 \pm \beta_6$ and such that the number of minus signs in β'_2, β'_3 and β'_4 is even, as well as all the other subsystems obtained by permuting the subscripts $\{1, \ldots, 6\}$ of the vectors β_j.

By Lemma 1.14 (i), the Gram matrix $(\langle \beta'_i, \beta'_j \rangle)_{i,j}$ is either $\frac{1}{4}\mathrm{id}_4$ or the matrix M_0 defined by (1.8). We further show that the second case can not occur. Indeed, if this were the case, then at least two of the norms $\|\beta'_2\|^2$, $\|\beta'_3\|^2$ and $\|\beta'_4\|^2$ are equal and after reordering, rescaling and sign change of the vectors β_j we may assume $\beta'_2 = \beta_1 + \beta_2$, $\beta'_3 = \beta_3 + \beta_4$, $\beta'_4 = \beta_5 + \beta_6$ and $\|\beta'_3\|^2 = \|\beta'_4\|^2 = \frac{1}{8}$, $\langle \beta'_3, \beta'_4 \rangle = \frac{1}{16}$. If we denote by $\beta''_j := \beta'_j$, $j = 1, 2$ and $\beta''_3 = \beta_3 - \beta_4$, $\beta''_4 = \beta_5 - \beta_6$, then $\{\sum_{j=1}^{4} \varepsilon_j \beta''_j\} \subset \mathcal{W}$ is also a subsystem of roots. Since $\{\|\beta''_1\|^2, \|\beta''_2\|^2\} = \{\frac{1}{4}, \frac{1}{8}\}$, it follows again by Lemma 1.14 (i) that the Gram matrix $(\langle \beta''_i, \beta''_j \rangle)_{i,j}$ is equal to M_0 defined by (1.8), so that $\|\beta''_4\|^2 = \|\beta''_3\|^2 = \frac{1}{8} = \|\beta'_3\|^2 = \|\beta'_4\|^2$. Thus, $\|\beta_1 + \beta_2\|^2 = \frac{1}{4}$, $\|\beta_{2j-1} + \beta_{2j}\|^2 = \frac{1}{8}$ for $j = 2, 3, 4$, and

$$\langle \beta_3 + \beta_4, \beta_5 + \beta_6 \rangle = \frac{1}{16}. \tag{1.24}$$

For every $3 \leq j < k \leq 8$, there exists $l \in \{2, 3, 4\}$ such that $\{2l-1, 2l\} \cap \{j, k\} = \emptyset$. Let $\{s, t\}$ denote the complement of $\{1, 2, j, k, 2l-1, 2l\}$ in $\{1, \ldots, 8\}$. The Gram matrix of $\{\beta_1 + \beta_2, \beta_{2l-1} + \beta_{2l}, \beta_j - \beta_k, \beta_s - \beta_t\}$ has at least two different values on the diagonal. By Lemma 1.14 (i) again, the remaining diagonal terms $\|\beta_j - \beta_k\|^2$ and $\|\beta_s - \beta_t\|^2$ are both equal to $\frac{1}{8}$. Thus $\|\beta_j - \beta_k\|^2 = \frac{1}{8}$ for $3 \leq j < k \leq 8$, and similarly $\|\beta_j + \beta_k\|^2 = \frac{1}{8}$ for $3 \leq j < k \leq 8$. Thus $\langle \beta_j, \beta_k \rangle = 0$ for all $3 \leq j < k \leq 8$, contradicting (1.24).

It then follows that the Gram matrix of any subsystem of roots $\{\sum_{j=1}^{4} \varepsilon_j \beta'_j\}$ as above is $(\langle \beta'_i, \beta'_j \rangle)_{i,j} = \frac{1}{4}\mathrm{id}_4$. In particular, this shows that for any $1 \leq i < j \leq 6$, $\|\beta_i + \beta_j\|^2 = \|\beta_i - \beta_j\|^2 = \frac{1}{4}$, and $\langle \alpha_1, \beta_i + \beta_j \rangle = \langle \alpha, \beta_i - \beta_j \rangle = 0$, implying that $\langle \beta_i, \beta_j \rangle = 0$ and $\langle \beta_i, \alpha_1 \rangle = 0$. For any $k \neq i, j$ we also have $\|\beta_i + \beta_k\|^2 = \|\beta_j + \beta_k\|^2 = \frac{1}{4}$, which then yields $\|\beta_i\|^2 = \frac{1}{8}$ for $1 \leq i \leq 6$. Denoting by $\beta' := \sum_{i=1}^{4} \beta'_i$, we compute $\langle \beta' - 2\beta'_i, \beta' \rangle = \frac{1}{2}$, for $1 \leq i \leq 4$. By (1.3) we obtain the new roots $\pm 2\beta'_i \in \overline{\mathcal{W}} \setminus \mathcal{W}$. Since the argument is true for any such subsystem of roots, we have the new roots $\{\pm 2\alpha_1, \pm 2(\beta_i \pm \beta_j) \,|\, 1 \leq i < j \leq 6\} \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$, which, by the above orthogonality relations, build the system of roots of $\mathfrak{so}(12) \oplus \mathfrak{su}(2)$. It thus follows that $\mathfrak{so}(12) \oplus \mathfrak{su}(2) \subseteq \mathfrak{h}$. Furthermore, $\mathcal{R}(\mathfrak{so}(12) \oplus \mathfrak{su}(2)) \cup \mathcal{W} = \mathcal{R}(\mathfrak{e}_7)$ (cf. [2, p. 56]), showing that $\mathfrak{e}_7 \subseteq \mathfrak{g}$.

We claim that $p = 1$. Assuming that $p \geq 2$, we consider $\alpha_2 \in \mathcal{A} \setminus \{\pm\alpha_1\}$. The previous argument also shows that $\langle \beta_i, \alpha_2 \rangle = 0$ for all $1 \leq i \leq 6$. Denoting by $a := \langle \alpha_1, \alpha_2 \rangle$, the scalar products between $\beta := \alpha_1 + \sum_{j=1}^{6} \beta_j$ and $\alpha_2 \pm (\beta_1 + \beta_2) \pm (\beta_3 + \beta_4) \pm (\beta_5 + \beta_6)$ take four possible values: $\{a \pm \frac{3}{4}, a \pm \frac{1}{4}\}$, thus contradicting (1.1), which only allows 3 different values for these scalar products. Hence, $p = 1$, and similarly $p' \leq 1$.

We further prove that $p' = 0$. Assuming the contrary, there exists $\gamma_1 \neq 0$ in $\mathcal{G}$. The same arguments as before show that $\langle \beta_i, \gamma_1 \rangle = 0$ for $1 \leq i \leq 6$ and $\|\gamma_1\|^2 = \frac{1}{4}$. Denoting by $a := \langle \alpha_1, \alpha_2 \rangle$, the set of scalar products between the unit vectors $\alpha_1 + \sum_{j=1}^{6} \beta_j$ and $\gamma_1 + \beta_1 - \beta_2 \pm (\beta_3 + \beta_4) \pm (\beta_5 + \beta_6)$ equals $\{a, a \pm \frac{1}{2}\}$. By (1.1), one has necessarily

$a = 0$, *i.e.* $\alpha_1 \perp \gamma_1$. We then denote by $\beta_7 := \frac{1}{2}(\alpha_1 + \beta_1)$ and $\beta_8 := \frac{1}{2}(\alpha_1 - \beta_1)$. The above relations show that $\langle \beta_i, \beta_j \rangle = 0$, $\|\beta_i\|^2 = \frac{1}{8}$ for $1 \le i < j \le 8$ and

$$\mathcal{W}(\mathfrak{m}) = \left\{ \sum_{j=1}^{8} \varepsilon_j \beta_j | \prod_{j=1}^{8} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_8}.$$

Let $\beta := \sum_{i=1}^{8} \beta_i$. Since $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \frac{1}{2}$ for $i \neq j$ and $\langle \beta - 2\beta_i - 2\beta_k, \beta - 2\beta_j - 2\beta_k \rangle = \frac{1}{2}$ for $i \neq j \neq k \neq i$, by (1.3) we obtain that $\{\pm 2(\beta_i \pm \beta_j) | 1 \le i < j \le 8\} \in \overline{\mathcal{W}} \setminus \mathcal{W} \subset \mathcal{R}(\mathfrak{h})$. This is exactly the system of roots of $\mathfrak{so}(16)$. Thus, $\mathfrak{so}(16) \subseteq \mathfrak{h}$. Since $\mathfrak{so}(16)$ is simple, the restriction to $\mathfrak{so}(16)$ of the Clifford morphism $\rho_* : \mathfrak{h} \to \mathfrak{so}(12)$ must vanish. Moreover, the restriction to $\mathfrak{so}(16)$ of the representations $\lambda_\pm$ from Lemma 1.11 vanish, too. Indeed, $p = p' = 1$ and $\mathbb{K} = \mathbb{H}$ so their complex dimensions equal 2. Thus the isotropy representation of G/H would vanish on $\mathfrak{so}(16) \subset \mathfrak{h}$, a contradiction.

As $p' = 0$, Lemma 1.11 shows that the isotropy representation can be written $\mathfrak{m} = \mu_+ \otimes_{\mathbb{H}} \lambda_+$, so as in (1.5) we can write

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{1.25}$$

with $\mathfrak{h}_1 := \ker(\rho_*)$, $\mathfrak{h}_2 := \ker(\lambda_+)$ and $\mathfrak{h}_0 = (\mathfrak{h}_1 \oplus \mathfrak{h}_2)^\perp$. Since $p = 1$, it follows that $\mathfrak{h}_0 \oplus \mathfrak{h}_1 \subseteq \mathfrak{su}(2)$. On the other hand, $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(12)$ and we have proved that $\mathfrak{so}(12) \oplus \mathfrak{su}(2) \subseteq \mathfrak{h}$. Hence, we obtain $\mathfrak{h}_2 = \mathfrak{so}(12)$, $\mathfrak{h}_0 = 0$ and $\mathfrak{h}_1 = \mathfrak{su}(2)$. In particular, we have $\mathfrak{h} = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$, and $\mathcal{R}(\mathfrak{g}) = \mathcal{W}(\mathfrak{m}) \oplus \mathcal{R}(\mathfrak{h})$ is isometric to the root system of $\mathfrak{e}_7$ (cf. [2, p. 56]). We conclude that $M = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$ by [2, Thm 6.1].

(IV) For $r = 16 = 2q$, it follows from Proposition 1.12 (IV) and Proposition 1.13 (III)-(IV) (for the extremal case $q = 8$) that (up to a sign change for one of the vectors β_i) $p = 1$, $p' = 0$ and the set of weights of the isotropy representation is given by:

$$\mathcal{W}(\mathfrak{m}) = \left\{ \sum_{j=1}^{8} \varepsilon_j \beta_j | \prod_{j=1}^{8} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_8}.$$

By Lemma 1.14 (b) it follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Thus, all roots in $\mathcal{W}$ have norm 1 and $\langle \beta, \beta_i \rangle = \frac{1}{8}$, where $\beta := \sum_{i=1}^{8} \beta_i$. This yields the following values for the scalar products for all i, j and k mutually distinct : $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \langle \beta - 2\beta_i - 2\beta_j, \beta - 2\beta_i - 2\beta_k \rangle = \frac{1}{2}$ and all other scalar products of roots in $\mathcal{W}$ are 0. The new roots we obtain by (1.3) are then $\{\pm 2(\beta_i \pm \beta_j) | 1 \le i < j \le 8\}$, which build the system of roots of $\mathfrak{so}(16)$. Thus, $\mathfrak{so}(16) \subseteq \mathfrak{h}$. As $p = 1$ and $p' = 0$ and $\mathfrak{so}(16) \subseteq \mathfrak{h}$, it follows with the notation from (1.25) that $\mathfrak{so}(16) \subseteq \mathfrak{h}_2$. On the other hand, ρ_* maps $\mathfrak{h}_0 \oplus \mathfrak{h}_2$ one-to-one into $\mathfrak{so}(16)$ and thus we must have equality: $\mathfrak{h}_0 \oplus \mathfrak{h}_2 = \mathfrak{so}(16)$. Consequently, $\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{so}_{16}) \cup \mathcal{W}$ is isometric to the system of roots of $\mathfrak{e}_8$ (cf. [2, p. 56]), showing that $\mathfrak{g} = \mathfrak{e}_8$ and $M = \mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ by [2, Thm 6.1]. $\square$

As already mentioned in Section 1, similar methods could be used to examine the remaining cases $r \le 8$. However, the arguments tend to be much more intricate as fast as the rank decreases. In order to keep this paper at a reasonable length, we have thus decided to limit our study to the extremal cases of Theorem 1.15.

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Chapter 2

Homogeneous almost quaternion-Hermitian manifolds

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Abstract. An almost quaternion-Hermitian structure on a Riemannian manifold (M^{4n}, g) is a reduction of the structure group of M to $\mathrm{Sp}(n)\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$. In this paper we show that a compact simply connected homogeneous almost quaternion-Hermitian manifold of non-vanishing Euler characteristic is either a Wolf space, or $\mathbb{S}^2 \times \mathbb{S}^2$, or the complex quadric $\mathrm{SO}(7)/\mathrm{U}(3)$.

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2.1 Introduction

The notion of (even) Clifford structures on Riemannian manifolds was introduced in [12]. Roughly speaking, a rank r (even) Clifford structure on M is a rank r Euclidean bundle whose (even) Clifford algebra bundle acts on the tangent bundle of M. For $r = 3$, an even Clifford structure on M is just an almost quaternionic structure, i.e. a rank 3 sub-bundle Q of the endomorphism bundle $\mathrm{End}\,(\mathrm{T}M)$ locally spanned by three endomorphisms I, J, K satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -\mathrm{id}, \qquad IJ = K.$$

If moreover $Q \subset \mathrm{End}^{-}(\mathrm{T}M)$ (or, equivalently, if I, J, K are g-orthogonal), the structure (M, g, Q) is called almost quaternion-Hermitian [7, 8, 9, 17].

Homogeneous even Clifford structures on homogeneous compact manifolds of non-vanishing Euler characteristic were studied in [11], where it is established an upper bound

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for their rank, as well as a description of the limiting cases. In this paper we consider the other extremal case, namely even Clifford structures with the lowest possible (non-trivial) rank, which is 3 and give the complete classification of compact homogeneous almost quaternion-Hermitian manifolds G/H with non-vanishing Euler characteristic. This last assumption turns out to be crucial at several places throughout the proof (see below). Without it, the classification is completely out of reach, but there are lots of homogeneous examples constructed for instance by D. Joyce [4, 5] and O. Maciá [6].

Our classification result is the following:

Theorem 2.1 *A compact simply connected homogeneous manifold $M = G/H$ of non-vanishing Euler characteristic carries a homogeneous almost quaternion-Hermitian structure if and only if it belongs to the following list:*

- *Wolf spaces G/N where G is any compact simple Lie group and N is the normalizer of some subgroup* $\mathrm{Sp}(1) \subset G$ *determined by a highest root of G, cf. [18].*
- $\mathbb{S}^2 \times \mathbb{S}^2$.
- $\mathrm{SO}(7)/\mathrm{U}(3)$.

Let us first give some comments on the above list. The Wolf spaces are quaternion-Kähler manifolds [18], so they admit not only a topological but even a *holonomy* reduction to $\mathrm{Sp}(n)\mathrm{Sp}(1)$. In dimension 4, every orientable manifold is almost quaternion-Hermitian since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$. In this dimension there exist (up to homothety) only two compact simply connected homogeneous manifolds with non-vanishing Euler characteristic: $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^4$. The latter is already a Wolf space since $\mathbb{S}^4 = \mathbb{H}\mathrm{P}^1$, this is why in dimension 4, the only extra space in the list is $\mathbb{S}^2 \times \mathbb{S}^2$. Finally, the complex quadric $\mathrm{SO}(7)/\mathrm{U}(3) \subset \mathbb{C}\mathrm{P}^7$, which incidentally is also the twistor space of $\mathbb{S}^6$, carries a 1-parameter family of $\mathrm{Sp}(3)\mathrm{U}(1)$ structures with fixed volume. Motivated by our present classification, F. Martín Cabrera and A. Swann [10] are currently investigating the quaternion Hermitian type of this family.

The outline of the proof of Theorem 2.1 is as follows: The first step is to show (in Proposition 2.6) that G has to be a simple Lie group, unless $M = \mathbb{S}^2 \times \mathbb{S}^2$. The condition $\chi(M) \neq 0$ (which is equivalent to $\mathrm{rk}(H) = \mathrm{rk}(G)$) is used here in order to ensure that every subgroup of maximal rank of a product $G_1 \times G_2$ is itself a product. The next step is to rule out the case $G = \mathcal{G}_2$ which is the only simple group for which the ratio between the length of the long and short roots is $\sqrt{3}$. Once this is done, we can thus assume that either all roots of G have the same length, or the ratio between the length of the long and short roots is $\sqrt{2}$. We further show that if G/H is symmetric, then H has an $\mathrm{Sp}(1)$-factor, so M is a Wolf space.

Now, since $\mathrm{rk}(H) = \mathrm{rk}(G)$, the weights of the (complexified) isotropy representation $\mathfrak{m}^{\mathbb{C}}$ can be identified with a subset of the root system of G. We show that the existence of a homogeneous almost quaternion-Hermitian structure on G/H implies that the set of weights $\mathfrak{W}(\mathfrak{m}^{\mathbb{C}})$ can be split into two distinct subsets, one of which is obtained from the other by a translation (Proposition 2.4 below). Moreover, if G/H is not symmetric, then $[\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{m} \neq 0$, so $(\mathfrak{W}(\mathfrak{m}^{\mathbb{C}}) + \mathfrak{W}(\mathfrak{m}^{\mathbb{C}})) \cap \mathfrak{W}(\mathfrak{m}^{\mathbb{C}}) \neq \emptyset$. Putting all this information together we are then able to show, using the properties of root systems, that there is one single isotropy weight system satisfying these conditions, namely the isotropy representation of $\mathrm{SO}(7)/\mathrm{U}(3)$, whose restriction to $\mathrm{SU}(3)$ is isomorphic to $\mathbb{C}^3 \oplus (\mathbb{C}^3)^*$ and is therefore quaternionic.

2.2 Preliminaries

Let $M = G/H$ be a homogeneous space. Throughout this paper we make the following assumptions:

- M is compact (and thus G and H are compact, too).
- The infinitesimal isotropy representation is faithful (this is always the case after taking an appropriate quotient of G)
- M has non-vanishing Euler characteristic: $\chi(M) \neq 0$, or, equivalently, $\mathrm{rk}(H) = \mathrm{rk}(G)$.
- M is simply connected. An easy argument using the exact homotopy sequence shows that by changing the representation of M as homogeneous space if necessary, one can assume that G is simply connected and H is connected (see [13] for example).

Denote by $\mathfrak{h}$ and $\mathfrak{g}$ the Lie algebras of H and G and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to some $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product on $\mathfrak{g}$. The restriction to $\mathfrak{m}$ of this scalar product defines a homogeneous Riemannian metric g on M.

An almost quaternion-Hermitian structure on a Riemannian manifold (M, g) is a three-dimensional sub-bundle of the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$, which is locally spanned by three endomorphisms satisfying the quaternion relations [7, 17]. In the case where $M = G/H$ is homogeneous, such a structure is called homogeneous if this three-dimensional sub-bundle is defined by a three-dimensional H-invariant summand of the second exterior power of the isotropy representation $\Lambda^2\mathfrak{m} = \mathrm{End}^-(\mathfrak{m})$. For our purposes, we give the following equivalent definition, which corresponds to the fact that an almost quaternion-Hermitian structure is just a rank 3 even Clifford structure (cf. [11, 12]):

Definition 2.2 *A* homogeneous almost quaternion-Hermitian *structure on the Riemannian homogeneous space $(G/H, g)$ is an orthogonal representation $\rho : H \to \mathrm{SO}(3)$ and an H-equivariant Lie algebra morphism $\varphi : \mathfrak{so}(3) \to \mathrm{End}^-(\mathfrak{m})$ extending to an algebra representation of the even real Clifford algebra Cl_3^0 on $\mathfrak{m}$.*

The H-equivariance of the morphism $\varphi : \mathfrak{so}(3) \to \mathrm{End}^-(\mathfrak{m})$ is with respect to the following actions of H: the action on $\mathfrak{so}(3)$ is given by the composition of the adjoint representation of $\mathrm{SO}(3)$ with ρ, and the action on $\mathrm{End}^-(\mathfrak{m})$ is the one induced by the isotropy representation ι of H. Since φ extends to a representation of $\mathrm{Cl}_3^0 \simeq \mathbb{H}$ on $\mathfrak{m}$, the above definition readily implies the following result (see also [11, Lemma 3.2] or [15]):

Lemma 2.3 *The complexified isotropy representation ι_* on $\mathfrak{m}^{\mathbb{C}}$ is isomorphic to the tensor product $\mathfrak{m}^{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{C}} \mathrm{E}$, where $\mathbb{H}$ is defined by the composition $\mu := \xi \circ \rho_*$ of ρ_* with the spin representation ξ of $\mathfrak{so}(3) = \mathfrak{spin}(3) = \mathfrak{sp}(1)$ on $\mathbb{H}$, and E is defined by the composition $\lambda := \pi \circ \iota_*$ of the isotropy representation with the projection of $\mathfrak{h}$ to the kernel of ρ_*.*

2.3 The classification

In this section we classify all compact simply connected homogeneous almost quaternion-Hermitian manifolds $M = G/H$ with non-vanishing Euler characteristic.

We choose a common maximal torus of H and G and denote by $\mathfrak{t} \subset \mathfrak{h}$ its Lie algebra. Then the root system $\mathcal{R}(\mathfrak{g}) \subset \mathfrak{t}^*$ is the disjoint union of the root system $\mathcal{R}(\mathfrak{h})$ and the set $\mathcal{W}$ of weights of the complexified isotropy representation of the homogeneous space G/H. This follows from the fact that the isotropy representation is given by the restriction to H of the adjoint representation of $\mathfrak{g}$.

The weights of the complex spin representation of $\mathfrak{so}(3)$ on $\Sigma_3^{\mathbb{C}} \simeq \mathbb{H}$ are $\mathfrak{W}(\Sigma_3^{\mathbb{C}}) = \left\{\pm\frac{1}{2}e_1\right\}$, where e_1 is some element of norm 1 of the dual of some Cartan sub-algebra of $\mathfrak{so}(3)$. We denote by $\beta \in \mathfrak{t}^*$ the pull-back through μ of the vector $\frac{1}{2}e_1$ and by $\mathcal{A} := \{\pm\alpha_1, \ldots, \pm\alpha_n\} \subset \mathfrak{t}^*$ the weights of the self-dual representation λ. By Lemma 2.3, we obtain the following description of the weights of the isotropy representation of any homogeneous almost quaternion-Hermitian manifold $M = G/H$, which is a particular case of [11, Proposition 3.3]:

Proposition 2.4 *The set $\mathcal{W} := \mathcal{W}(\mathfrak{m})$ of weights of the isotropy representation is given by:*

$$\mathcal{W} = \{\varepsilon_i\alpha_i + \varepsilon\beta\}_{1\leq i\leq n;\varepsilon_i,\varepsilon\in\{\pm 1\}}. \tag{2.1}$$

As an immediate consequence we have:

Lemma 2.5 *Let $(G/H, g, \rho, \varphi)$ be a homogeneous almost quaternion-Hermitian structure as in Definition 2.2. Then the infinitesimal representation $\rho_* : \mathfrak{h} \to \mathfrak{so}(3)$ does not vanish.*

Proof. Suppose for a contradiction that $\rho_* = 0$. Then the $\mathfrak{h}$-representation $\mathbb{H}$ defined in Lemma 2.3 is trivial, so $\beta = 0$ and $\mathfrak{m}^{\mathbb{C}} = \mathrm{E} \oplus \mathrm{E}$. Every weight of the (complexified) isotropy representation appears then twice in the root system of G, which is impossible (*cf.* [16, p. 38]). $\square$

Our next goal is to show that the automorphism group of a homogeneous almost quaternion-Hermitian manifold is in general a simple Lie group:

Proposition 2.6 *If G/H is a simply connected compact homogeneous almost quaternion-Hermitian manifold with non-vanishing Euler characteristic, then either G is simple or $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $M = \mathbb{S}^2 \times \mathbb{S}^2$.*

Proof. We already know that G is compact and simply connected. If G is not simple, then $G = G_1 \times G_2$ with $\dim(G_i) \geq 3$. Let $\mathfrak{g}_i$ denote the Lie algebra of G_i, so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. By a classical result of Borel and Siebenthal ([3, p. 210]), the Lie algebra of the subgroup H splits as $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. Correspondingly, the isotropy representation splits as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_i$ is the isotropy representation of $\mathfrak{h}_i$ in $\mathfrak{g}_i$.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathfrak{so}(3)$ and let us denote by $J_i := \varphi(e_i)$, for $1 \leq i \leq 3$. The H-equivariance of φ implies that

$$\varphi(\rho_*(X)e_i) = [\mathrm{ad}_X, J_i], \qquad \forall\, X \in \mathfrak{h},\ 1 \leq i \leq 3. \tag{2.2}$$

We claim that the representation ρ_* does not vanish on $\mathfrak{h}_1$ or on $\mathfrak{h}_2$. Assume for instance that $\rho_*(\mathfrak{h}_1) = 0$. We express each endomorphism J_i of $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ as

$$J_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}.$$

For every $X \in \mathfrak{h}_1$, (2.2) shows that ad_X commutes with J_i. Expressing

$$\mathrm{ad}_X = \begin{pmatrix} \mathrm{ad}_X^{\mathfrak{g}_1} & 0 \\ 0 & 0 \end{pmatrix}$$

we get in particular that $\mathrm{ad}_X^{\mathfrak{g}_1} \circ B_i = 0$ for all $X \in \mathfrak{h}_1$. On the other hand, since $\mathrm{rk}(\mathfrak{h}_1) = \mathrm{rk}(\mathfrak{g}_1)$, there exists no vector in $\mathfrak{m}_1$ commuting with all $X \in \mathfrak{h}_1$, so $B_i = 0$ and thus $C_i = -B_i^* = 0$ for $1 \leq i \leq 3$. However, this would imply that the map $\varphi_1 : \mathfrak{so}(3) \to \mathrm{End}^-(\mathfrak{m}_1)$ given by $\varphi_1(e_i) = A_i$ for $1 \leq i \leq 3$ is a homogeneous almost quaternion-Hermitian structure on G_1/H_1 with vanishing ρ_*, which contradicts Lemma 2.5. This proves our claim.

Now, since $\rho_* : \mathfrak{h} \to \mathfrak{so}(3)$ is a Lie algebra morphism, we must have in particular

$$[\rho_*(\mathfrak{h}_1), \rho_*(\mathfrak{h}_2)] = 0.$$

By changing the orthonormal basis $\{e_1, e_2, e_3\}$ if necessary, we thus may assume that $\rho_*(\mathfrak{h}_1) = \rho_*(\mathfrak{h}_2) = \langle e_1 \rangle$. The Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ decompose as $\mathfrak{h}_i = \mathfrak{h}_i' \oplus \langle X_i \rangle$ where $\mathfrak{h}_i' := \ker(\rho_*) \cap \mathfrak{h}_i$ and $\rho_*(X_i) = e_1$ for $1 \leq i \leq 2$.

From (2.2), the following relations hold:

$$[\mathrm{ad}_{X_i}, J_2] = J_3, \quad [\mathrm{ad}_{X_i}, J_3] = -J_2, \qquad 1 \leq i \leq 2. \tag{2.3}$$

Like before we can write

$$\mathrm{ad}_{X_1} = \begin{pmatrix} \mathrm{ad}_{X_1}^{\mathfrak{g}_1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{ad}_{X_2}^{\mathfrak{g}_2} \end{pmatrix},$$

so (2.3) implies that $A_2 = A_3 = 0$ and $D_2 = D_3 = 0$. In particular

$$-1 = J_2^2 = \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix}^2 = \begin{pmatrix} B_2C_2 & 0 \\ 0 & C_2B_2 \end{pmatrix},$$

thus showing that B_2 defines an isomorphism between $\mathfrak{m}_2$ and $\mathfrak{m}_1$ (whose inverse is $-C_2$).

On the other hand, since by (2.2) ad_X commutes with J_2 for all $X \in \mathfrak{h}_1'$, we obtain as before that $\mathrm{ad}_X^{\mathfrak{g}_1} \circ B_2 = 0$ for all $X \in \mathfrak{h}_1'$. Since B_2 is onto, this shows that the isotropy representation of G_1/H_1 restricted to $\mathfrak{h}_1'$ vanishes, so $\mathfrak{h}_1' = 0$ and similarly $\mathfrak{h}_2' = 0$. We therefore have $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathbb{R}$, and since $\mathrm{rk}(G_i) = \mathrm{rk}(H_i) = 1$, we get $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{su}(2)$. We thus have $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and $H = \mathbb{T}^2$ is a maximal torus, so $M = \mathbb{S}^2 \times \mathbb{S}^2$. □

We are in position to complete the proof of our main result:

Proof. [Proof of Theorem 2.1] By Proposition 2.6 we may assume that G is simple. We first study the case $G = \mathcal{G}_2$ (this is the only simple group for which the ratio between the length of long and short roots is neither 1, nor $\sqrt{2}$). The only connected subgroups of rank 2 of $\mathcal{G}_2$ are $\mathrm{U}(2), \mathrm{SU}(3), \mathrm{SO}(4)$ and $\mathbb{T}^2$. The spaces $\mathcal{G}_2/\mathrm{U}(2)$ and $\mathcal{G}_2/\mathrm{SU}(3)$ have dimension 10 and 6 respectively, therefore they can not carry almost quaternion-Hermitian structures.

The quotient $\mathcal{G}_2/\mathrm{SO}(4)$ is a Wolf space, so it remains to study the generalized flag manifold $\mathcal{G}_2/\mathbb{T}^2$. We claim that this space has no homogeneous almost quaternion-Hermitian structure. Indeed, if this were the case, using Proposition 2.4 one could express the root system of $\mathcal{G}_2$ as the disjoint union of two subsets

$$\mathfrak{W}^+ := \{\varepsilon_i\alpha_i + \beta\}_{1\le i\le 3;\varepsilon_i\in\{\pm1\}}, \qquad \mathfrak{W}^- := \{\varepsilon_i\alpha_i - \beta\}_{1\le i\le 3;\varepsilon_i\in\{\pm1\}}$$

such that there exists some vector v $(:= 2\beta)$ with $\mathfrak{W}^+ = v + \mathfrak{W}^-$. On the other hand, it is easy to check that there exist no such partition of $\mathbb{R}(\mathcal{G}_2)$.

Consider now the case where $M = G/H$ is a symmetric space. If M is a Wolf space there is nothing to prove, so assume from now on that this is not the case. The Lie algebra of H can be split as $\mathfrak{h} = \ker(\rho_*) \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ denotes the orthogonal complement of $\ker(\rho_*)$. Clearly $\mathfrak{h}_0$ is isomorphic to $\rho_*(\mathfrak{h}) \subset \mathfrak{so}(3)$ so by Lemma 2.5, $\mathfrak{h}_0 = \mathfrak{u}(1)$ or $\mathfrak{h}_0 = \mathfrak{sp}(1)$. The latter case can not occur since our assumption that M is not a Wolf space implies that $\mathfrak{h}$ has no $\mathfrak{sp}(1)$-summand. We are left with the case when $\mathfrak{h} = \ker(\rho_*) \oplus \mathfrak{u}(1)$. We claim that this case can not occur either. Indeed, if such a space would carry a homogeneous almost quaternion-Hermitian structure, then the representation of $\ker(\rho_*)$ on $\mathfrak{m}$ would be quaternionic. Two anti-commuting complex structures I, J of $\mathfrak{m}$ induce non-vanishing elements a_I, a_J in the center of $\ker(\rho_*)$ (see the proof of [14, Lemma 2.4]). On the other hand, the adjoint actions of a_I and a_J on $\mathfrak{m}$ are proportional to I and J respectively ([14, Eq. (4)]) and thus anti-commute, contradicting the fact that a_I and a_J commute (being central elements).

We can assume from now on, that $M = G/H$ is non-symmetric, G is simple and $G \neq \mathcal{G}_2$. Up to a rescaling of the ad_G-invariant metric on $\mathfrak{g}$, we may thus assume that all roots of $\mathfrak{g}$ have square length equal to 1 or 2.

From (2.1), it follows that

$$\mathcal{R}(\mathfrak{g}) = \mathcal{W}(\mathfrak{m}) \cup \mathcal{R}(\mathfrak{h}) = \{\varepsilon_i\alpha_i + \varepsilon\beta\}_{1\le i\le n;\varepsilon_i,\varepsilon\in\{\pm1\}} \cup \mathcal{R}(\mathfrak{h}).$$

Up to a change of signs of the α_i's, we may assume:

$$\langle\, \beta,\, \alpha_i \,\rangle \geq 0, \quad \text{for all } 1 \leq i \leq n. \tag{2.4}$$

Then either the roots $\beta+\alpha_i$ and $\beta-\alpha_i$ of G have the same length, or $|\beta+\alpha_i|^2=2$ and $|\beta-\alpha_i|^2=1$. This shows that for each $1\le i\le n$,

$$\langle\,\beta,\,\alpha_i\,\rangle\in\left\{0,\frac{1}{4}\right\}. \tag{2.5}$$

From the general property of root systems (2.10) below, it follows that:

$$|\beta|^2\in\left\{\frac{1}{4},\frac{3}{4},\frac{5}{4}\right\}. \tag{2.6}$$

Since the homogeneous space G/H is not symmetric, we have $[\mathfrak{m},\mathfrak{m}]\not\subseteq\mathfrak{h}$, so there exist subscripts $i,j,k\in\{1,\dots,n\}$ such that $(\pm\beta\pm\alpha_i)+(\pm\beta\pm\alpha_j)=\pm\beta\pm\alpha_k$. Taking (2.5) into account, we need to check the following possible cases (up to a permutation of the subscripts):

a) $\beta=\pm2\alpha_1\pm\alpha_2$.

b) $\beta=\frac{\alpha_1}{3}$.

c) $\beta=\frac{\alpha_1\pm\alpha_2\pm\alpha_3}{3}$.

d) $\beta=\alpha_1\pm\alpha_2\pm\alpha_3$.

We will show that cases a), b) and c) can not occur and that in case d) there is only one solution.

a) If $\beta=2\alpha_1+\alpha_2$, then $\beta+\alpha_2=2(\beta-\alpha_1)$ and this would imply the existence of two proportional roots, $\beta+\alpha_2$ and $\beta-\alpha_1$, in $\mathcal{W}\subseteq\mathcal{R}(\mathfrak{g})$, contradicting the property R2 of root systems (*cf.* Definition 2.7). For all the other possible choices of signs in a) we obtain a similar contradiction.

b) If $\beta=\frac{\alpha_1}{3}$, then there exist two proportional roots: $\beta+\alpha_1=-2(\beta-\alpha_1)$ in $\mathcal{R}(\mathfrak{g})$, which again contradicts R2.

c) If $\beta=\frac{\alpha_1\pm\alpha_2\pm\alpha_3}{3}$, then $|\beta|^2=\frac{1}{3}(\langle\,\beta,\,\alpha_1\,\rangle\pm\langle\,\beta,\,\alpha_2\,\rangle\pm\langle\,\beta,\,\alpha_3\,\rangle)$. From (2.5) and (2.6), it follows that the only possibility is:

$$\beta=\frac{\alpha_1+\alpha_2+\alpha_3}{3},|\beta|^2=\frac{1}{4}\text{ and }\langle\,\beta,\,\alpha_i\,\rangle=\frac{1}{4},\text{ for }1\le i\le 3.$$

Together with (2.10), this implies that for each $1\le i\le 3$ we have: $|\beta+\alpha_i|^2=2$, $|\beta-\alpha_i|^2=1$ and $|\alpha_i|^2=\frac{5}{4}$. Thus, for all $1\le i,j\le 3$, $i\neq j$, we have:

$$\langle\,\beta+\alpha_i,\,\beta-\alpha_j\,\rangle=\frac{1}{4}-\langle\,\alpha_i,\,\alpha_j\,\rangle,$$

which by (2.10) must be equal to 0 or ±1, showing that $\langle\,\alpha_i,\,\alpha_j\,\rangle\in\{-\frac{3}{4},\frac{1}{4},\frac{5}{4}\}$. On the other hand, a direct computation shows that

$$\langle\,\alpha_1,\,\alpha_2\,\rangle+\langle\,\alpha_1,\,\alpha_3\,\rangle+\langle\,\alpha_2,\,\alpha_3\,\rangle=\frac{1}{2}\left(9|\beta|^2-\frac{15}{4}\right)=-\frac{3}{4},$$

which is not possible for any of the above values of the scalar products, yielding a contradiction.

d) From (2.5) and (2.6), it follows that there are three possible sub-cases:

Case 1. $\beta = \alpha_1 \pm \alpha_2 \pm \alpha_3, |\beta|^2 = \frac{1}{4}, \langle \beta, \alpha_1 \rangle = \frac{1}{4}, \langle \beta, \alpha_2 \rangle = \langle \beta, \alpha_3 \rangle = 0.$

Case 2. $\beta = \alpha_1 + \alpha_2 - \alpha_3, |\beta|^2 = \frac{1}{4}, \langle \beta, \alpha_i \rangle = \frac{1}{4}, 1 \le i \le 3.$

Case 3. $\beta = \alpha_1 + \alpha_2 + \alpha_3, |\beta|^2 = \frac{3}{4}, \langle \beta, \alpha_i \rangle = \frac{1}{4}, 1 \le i \le 3.$

Case 1. From (2.10) it follows $|\alpha_1|^2 = \frac{5}{4}$ and $|\alpha_2|^2 = |\alpha_3|^2 = \frac{3}{4}$. Since $\langle \beta + \alpha_1, \beta - \alpha_1 \rangle = -1$ and $|\beta + \alpha_1|^2 = 2|\beta - \alpha_1|^2$, the reflexion property (2.11) shows that $2\beta = (\beta + \alpha_1) + (\beta - \alpha_1)$ and $3\beta - \alpha_1 = (\beta + \alpha_1) + 2(\beta - \alpha_1)$ belong to $\mathcal{R}(\mathfrak{g})$. We show that these roots actually belong to $\mathcal{R}(\mathfrak{h})$, i.e. that $2\beta, 3\beta - \alpha_1 \notin \mathcal{W}$. We argue by contradiction.

Let us first assume that $2\beta \in \mathcal{W}$. Then there exists k, $1 \le k \le n$, such that $2\beta = \pm\beta \pm \alpha_k$. If $\beta = \pm\alpha_k$ we obtain that $0 = \beta \mp \alpha_k$ belongs to $\mathbb{R}(\mathfrak{g})$, which contradicts the property R1 of root systems. If $\beta = \pm\frac{\alpha_k}{3}$, then the roots $\beta + \alpha_k$ and $\beta - \alpha_k$ are proportional, which contradicts R2.

Now we assume that $3\beta - \alpha_1 \in \mathcal{W}$ and conclude similarly. In this case there exists k, $1 \le k \le n$ such that either $2\beta = \alpha_1 \pm \alpha_k$ or $4\beta = \alpha_1 \pm \alpha_k$. In the first case we obtain $\beta - \alpha_1 = -\beta \pm \alpha_k$, which contradicts the fact that roots of G are simple. In the second case (2.5) yields $|\beta|^2 = \frac{1}{4}\langle \beta, \alpha_1 \rangle \pm \frac{1}{4}\langle \beta, \alpha_k \rangle \le \frac{1}{8}$, which contradicts (2.6).

This shows that $2\beta, 3\beta - \alpha_1 \in \mathcal{R}(\mathfrak{h})$. Moreover $\langle 2\beta, 3\beta - \alpha_1 \rangle = 1$ and thus, by (2.11), their difference is a root of $\mathfrak{h}$ too: $\beta - \alpha_1 = (3\beta - \alpha_1) - (2\beta) \in \mathcal{R}(\mathfrak{h})$, which is in contradiction with $\beta - \alpha_1 \in \mathcal{W}$. Consequently, case 1. can not occur.

Case 2. From (2.10) it follows that $|\alpha_i|^2 = \frac{5}{4}$, for all $1 \le i \le 3$. For all $1 \le i, j \le 3$, $i \ne j$, we then compute: $\langle \beta + \alpha_i, \beta + \alpha_j \rangle = \frac{3}{4} + \langle \alpha_i, \alpha_j \rangle$, which by (2.10) must be equal to 0 or ± 1, implying that $\langle \alpha_i, \alpha_j \rangle \in \{-\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}\}$. On the other hand, we obtain

$$\langle \alpha_1, \alpha_2 \rangle + \langle \alpha_1, \alpha_3 \rangle + \langle \alpha_2, \alpha_3 \rangle = \frac{1}{2}\left(|\beta|^2 - \frac{15}{4}\right) = -\frac{7}{4},$$

which is not possible for any of the above values of the scalar products, yielding again a contradiction.

Case 3. From (2.10) it follows that $|\alpha_i|^2 = \frac{3}{4}$, for all $1 \le i \le 3$. Computing the norm of $\beta - \alpha_k = \alpha_i + \alpha_j$, where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$, yields that $\langle \alpha_i, \alpha_j \rangle = -\frac{1}{4}$, for all $1 \le i, j \le 3$, $i \ne j$. We then get

$$\langle \beta + \alpha_i, \beta + \alpha_j \rangle = 1, \qquad \text{for all } 1 \le i, j \le 3, i \ne j,$$

which by the reflexion property (2.11) implies that

$$\{\alpha_i - \alpha_j\}_{1 \le i, j \le 3} \subseteq \mathcal{R}(\mathfrak{g}). \tag{2.7}$$

We claim that $n = 3$ (recall that n denotes the number of vectors α_i, or equivalently the quaternionic dimension of M). Assume for a contradiction that $n \geq 4$. By (2.5), $\langle \beta, \alpha_l \rangle = \frac{1}{4}$ or $\langle \beta, \alpha_l \rangle = 0$, for any $4 \leq l \leq n$.

If $\langle \beta, \alpha_l \rangle = \frac{1}{4}$ for some $l \geq 4$, it follows that $|\alpha_l|^2 = \frac{3}{4}$ and $|\beta + \alpha_l|^2 = 2$, implying by (2.10) that the scalar product $\langle \beta - \alpha_i, \beta + \alpha_l \rangle$ belongs to $\{\pm 1, 0\}$, for $1 \leq i \leq 3$. This further yields that $\langle \alpha_i, \alpha_l \rangle \in \{\frac{7}{4}, \frac{3}{4}, -\frac{1}{4}\}$. On the other hand, the Cauchy-Schwarz inequality applied to α_i and α_l and the fact that $\mathcal{W}$ has only simple roots (being a root sub-system) imply that the only possible value is $\langle \alpha_i, \alpha_l \rangle = -\frac{1}{4}$, for $1 \leq i \leq 3$ and $4 \leq l \leq n$. Thus, $|\beta + \alpha_l|^2 = 0$, which contradicts the property R1 of root systems (*cf.* Definition 2.7).

We therefore have $\langle \beta, \alpha_l \rangle = 0$, for all $4 \leq l \leq n$. If $|\beta \pm \alpha_l|^2 = 2$ for some $l \geq 4$ then $|\alpha_l|^2 = \frac{5}{4}$, so $\langle \beta - \alpha_l, \beta + \alpha_l \rangle = -\frac{1}{2}$, contradicting (2.10). Thus $|\beta \pm \alpha_l|^2 = 1$ for all $4 \leq l \leq n$. If $n \geq 5$, (2.10) implies

$$\langle \beta - \alpha_l, \beta + \alpha_s \rangle, \langle \beta - \alpha_l, \beta - \alpha_s \rangle \in \left\{0, \pm\frac{1}{2}\right\}, \qquad \text{for } 4 \leq l, s \leq n,\ l \neq s.$$

This contradicts the equality $\langle \beta - \alpha_l, \beta + \alpha_s \rangle + \langle \beta - \alpha_l, \beta - \alpha_s \rangle = \frac{3}{2}$, showing that $n \leq 4$.

It remains to show that the existence of $\alpha_4 \in \mathcal{A}$, which by the above necessarily satisfies $\langle \beta, \alpha_4 \rangle = 0$ and $|\alpha_4|^2 = \frac{1}{4}$, leads to a contradiction. By (2.10), it follows that

$$1 + \langle \alpha_i, \alpha_4 \rangle = \langle \beta + \alpha_i, \beta + \alpha_4 \rangle \in \{\pm 1, 0\}, \qquad \forall\, 1 \leq i \leq 3.$$

This constraint together with the Cauchy-Schwarz inequality, $|\langle \alpha_i, \alpha_4 \rangle| \leq \frac{\sqrt{3}}{4}$, implies that $\langle \alpha_i, \alpha_4 \rangle = 0$, for $1 \leq i \leq 3$.

Applying the reflexion property (2.11) to $\beta + \alpha_4$ and $\beta + \alpha_i$, for $1 \leq i \leq 3$, which satisfy $\langle \beta+\alpha_i, \beta+\alpha_4 \rangle = 1$ and $|\beta+\alpha_i|^2 = 2|\beta+\alpha_4|^2$, it follows that $\alpha_i - \alpha_4, \beta + 2\alpha_4 - \alpha_i \in \mathcal{R}(\mathfrak{g})$. We now show that all these roots actually belong to $\mathcal{R}(\mathfrak{h})$. Let us assume that $\alpha_i - \alpha_4 \in \mathcal{W}$ for some $i \leq 3$, i.e. there exists s, $1 \leq s \leq 4$, such that $\alpha_i - \alpha_4 \in \{\pm\beta \pm \alpha_s\}$. Since α_4 is orthogonal to β and to α_i, for $1 \leq i \leq 3$, it follows that $\alpha_i - \alpha_4$ must be equal to $\pm\beta - \alpha_4$, leading to the contradiction that $0 = \beta \mp \alpha_i \in \mathcal{W}$. Therefore $\alpha_i - \alpha_4 \in \mathcal{R}(\mathfrak{h})$. A similar argument shows that $\beta + 2\alpha_4 - \alpha_i \in \mathcal{R}(\mathfrak{h})$.

Now, since the scalar product of these two roots of $\mathfrak{h}$ is $\langle \alpha_i - \alpha_4, \beta + 2\alpha_4 - \alpha_i \rangle = -1$, it follows again by (2.11) that their sum $\beta + \alpha_4$ also belongs to $\mathcal{R}(\mathfrak{h})$, contradicting the fact that $\beta + \alpha_4 \in \mathcal{W}(\mathfrak{m})$. This finishes the proof of the claim that $n = 3$.

Since the determinant of the Gram matrix $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq 3}$ is equal to $\frac{5}{16}$, the vectors $\{\alpha_i\}_{1 \leq i \leq 3}$ are linearly independent. Thus the roots of $\mathfrak{g}$ given by (2.7) can not belong to $\mathcal{W}$, and therefore $\{\alpha_i - \alpha_j\}_{1 \leq i,j \leq 3}$ belong to $\mathcal{R}(\mathfrak{h})$.

Concluding, we have proven that $n = 3$ and that the following inclusions hold (after introducing the notation $\gamma_i := \alpha_j + \alpha_k$ for all permutations $\{i, j, k\}$ of $\{1, 2, 3\}$):

$$\{\gamma_i - \gamma_j\}_{1 \leq i \neq j \leq 3} \subseteq \mathcal{R}(\mathfrak{h}), \quad \{\gamma_i - \gamma_j\}_{1 \leq i \neq j \leq 3} \cup \{\pm\gamma_i\}_{1 \leq i \leq 3} \subseteq \mathcal{R}(\mathfrak{g}), \tag{2.8}$$

where $\langle \gamma_i, \gamma_j \rangle = \delta_{ij}$, for all $1 \leq i, j \leq 3$.

Since these sets are closed root systems and we are interested in the representation of M as a homogeneous space G/H with the smallest possible group G, we may assume that we have equality in (2.8). Hence $\mathcal{R}(\mathfrak{h}) = \{\gamma_i - \gamma_j\}_{1\leq i\neq j\leq 3}$, with $\{\gamma_i\}_{1\leq i\leq 3}$ an orthonormal basis, (which is exactly the root system of the Lie algebra $\mathfrak{su}(3)$), and $\mathcal{R}(\mathfrak{g}) = \{\gamma_i - \gamma_j\}_{1\leq i\neq j\leq 3} \cup \{\pm\gamma_i\}_{1\leq i\leq 3}$, which is the root system of $\mathfrak{so}(7)$. We conclude that the only possible solution is the simply connected homogeneous space SO(7)/U(3).

It remains to check that this space indeed carries a homogeneous almost quaternionic-Hermitian structure. Using the sequence of inclusions

$$\mathfrak{u}(3) \subset \mathfrak{so}(6) \subset \mathfrak{so}(7),$$

we see that the isotropy representation $\mathfrak{m}$ of SO(7)/U(3) is the direct sum of the restriction to $U(3)$ of the isotropy representation of the sphere SO(7)/SO(6), (which is just the standard representation of U(3) on $\mathbb{C}^3$), and of the isotropy representation of SO(6)/U(3), which is $\Lambda^2(\mathbb{C}^3)$ (cf. [2, p. 312]):

$$\mathfrak{m} = \mathbb{C}^3 \oplus \Lambda^2(\mathbb{C}^3).$$

Let I denote the complex structure of $\mathfrak{m}$. After identifying U(1) with the center of U(3) via the diagonal embedding, an element $z \in$ U(1) acts on $\mathfrak{m}$ by complex multiplication with z^3, i.e. $\iota(z) = z^3$. Since $\Lambda^2(\mathbb{C}^3) = (\mathbb{C}^3)^*$ as complex SU(3)-representations, it follows that the restriction to SU(3) of the isotropy representation $\mathfrak{m}$ is $\mathbb{C}^3 \oplus (\mathbb{C}^3)^*$, and thus carries a quaternionic structure, i.e. a complex anti-linear automorphism J. We claim that a homogeneous almost quaternionic-Hermitian structure on SO(7)/U(3) in the sense of Definition 2.2 is given by ρ : U(3) $\to$ SO(3) and $\varphi : \mathfrak{so}(3) \simeq \mathrm{Im}(\mathbb{H}) \to \mathrm{End}^-(\mathfrak{m})$ defined by

$$\rho(A) = \det(A), \qquad \varphi(i) = I,\ \varphi(j) = J,\ \varphi(k) = IJ,$$

where $\det(A) \in$ U(1) is viewed as an element in SO(3) via the composition

$$\mathrm{U}(1) = S(\mathbb{C}) \to S(\mathbb{H}) = \mathrm{Spin}(3) \to \mathrm{SO}(3).$$

Indeed, the only thing to check is the equivariance of φ, i.e.

$$\varphi(\rho(A)M\rho(A)^{-1}) = \iota(A)\varphi(M)\iota(A)^{-1}, \qquad \forall M \in \mathfrak{so}(3),\ \forall A \in \mathrm{U}(3). \tag{2.9}$$

Write $A = zB$ with $B \in$ SU(3). Then $\rho(A) = z^3$, $\iota(A) = z^3\iota(B)$ and $\iota(B)$ commutes with I, J, K, thus with $\varphi(M)$. The relation (2.9) is trivially satisfied for $M = i$, whereas for $M = j$ or $M = k$ one has $Mz = \bar{z}M = z^{-1}M$ and similarly $\varphi(M)\iota(z)\iota(B) = \iota(z^{-1})\iota(B)\varphi(M)$, so

$$\varphi(\rho(A)M\rho(A)^{-1}) = \varphi(z^3Mz^{-3}) = \varphi(z^6M) = z^6\varphi(M) = \iota(z^2)\varphi(M) = \iota(A)\varphi(M)\iota(A)^{-1}.$$

This finishes the proof of the theorem. $\square$

A Root systems

For the basic theory of root systems we refer to [1] and [16].

Definition 2.7 *A set $\mathcal{R}$ of vectors in a Euclidean space $(V, \langle \cdot, \cdot \rangle)$ is called a* root system *if it satisfies the following conditions:*

R1 *$\mathcal{R}$ is finite,* $\operatorname{span}(\mathcal{R}) = V$, $0 \notin \mathcal{R}$.

R2 *If $\alpha \in \mathcal{R}$, then the only multiples of α in $\mathcal{R}$ are $\pm\alpha$.*

R3 $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$, *for all $\alpha, \beta \in \mathcal{R}$.*

R4 $s_\alpha : \mathcal{R} \to \mathcal{R}$, *for all $\alpha \in \mathcal{R}$ (s_α is the reflection $s_\alpha : V \to V$, $s_\alpha(v) := v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle}\alpha$).*

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ endowed with an $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product. Fix a Cartan sub-algebra $\mathfrak{t} \subset \mathfrak{g}$ and let $\mathcal{R}(\mathfrak{g}) \subset \mathfrak{t}^*$ denote its root system. It is well-known that $\mathcal{R}(\mathfrak{g})$ satisfies the conditions in Definition 2.7. Conversely, every set of vectors satisfying the conditions in Definition 2.7 is the root system of a unique semi-simple Lie algebra of compact type.

Remark 2.8 (Properties of root systems) *Let $\mathcal{R}$ be a root system. If $\alpha, \beta \in \mathcal{R}$ such that $\beta \neq \pm\alpha$ and $\|\beta\|^2 \geq \|\alpha\|^2$, then either $\langle \alpha, \beta \rangle = 0$ or*

$$\left(\frac{\|\beta\|^2}{\|\alpha\|^2}, \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \right) \in \{(1, \pm 1), (2, \pm 2), (3, \pm 3)\}. \tag{2.10}$$

In other words, either the scalar product of two roots vanishes, or its absolute value equals half the square length of the longest root. Moreover,

$$\beta - \operatorname{sgn}\left(\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \right) k\alpha \in \mathcal{R}, \qquad \textit{for all } k \in \mathbb{Z}, 1 \leq k \leq \left| \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \right|. \tag{2.11}$$

Definition 2.9 ([11]) *A set $\mathcal{P}$ of vectors in a Euclidean space $(V, \langle \cdot, \cdot \rangle)$ is called a* root sub-system *if it satisfies the conditions R1 - R3 from Definition 2.7 and if the set $\overline{\mathcal{P}}$ obtained from $\mathcal{P}$ by taking all possible reflections is a root system.*

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Chapter 3

Eigenvalue Estimates of the Spin^c Dirac Operator and harmonic forms on Kähler-Einstein Manifolds

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Abstract. We establish a lower bound for the eigenvalues of the Dirac operator defined on a compact Kähler-Einstein manifold of positive scalar curvature and endowed with particular spin^c structures. The limiting case is characterized by the existence of Kählerian Killing spin^c spinors in a certain subbundle of the spinor bundle. Moreover, we show that the Clifford multiplication between an effective harmonic form and a Kählerian Killing spin^c spinor field vanishes. This extends to the spin^c case the result of A. Moroianu stating that, on a compact Kähler-Einstein manifold of complex dimension $4\ell + 3$ carrying a complex contact structure, the Clifford multiplication between an effective harmonic form and a Kählerian Killing spinor is zero.

1 Introduction

The geometry and topology of a compact Riemannian spin manifold (M^n, g) are strongly related to the existence of special spinor fields and thus, to the spectral properties of a fundamental operator called the Dirac operator D [1, 27]. A. Lichnerowicz [27] proved, under the weak condition of the positivity of the scalar curvature, that the kernel of the Dirac operator is trivial. Th. Friedrich [6] gave the following lower bound for the first eigenvalue λ of D on a compact Riemannian spin manifold (M^n, g):

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S, \tag{3.1}$$

where S denotes the scalar curvature, assumed to be nonnegative. Equality holds if and only if the corresponding eigenspinor φ is parallel (if $\lambda = 0$) or a Killing spinor of

Killing constant $-\frac{\lambda}{n}$ (if $\lambda \neq 0$), *i.e.* if $\nabla_X\varphi = -\frac{\lambda}{n}\ X \cdot \varphi$, for all vector fields X, where "$\cdot$" denotes the Clifford multiplication and ∇ is the spinorial Levi-Civita connection on the spinor bundle ΣM. Killing spinors force the underlying metric to be Einstein. The classification of complete simply-connected Riemannian spin manifolds with real Killing spinors was done by C. Bär [2]. Useful geometric information has been also obtained by restricting parallel and Killing spinors to hypersurfaces [3, 10, 11, 12, 13, 14]. O. Hijazi proved that the Clifford multiplication between a harmonic k-form β ($k \neq 0, n$) and a Killing spinor vanishes. In particular, the equality case in (3.1) cannot be attained on a Kähler spin manifold, since the Clifford multiplication between the Kähler form and a Killing spinor is never zero. Indeed, on a Kähler compact manifold (M^{2m}, g, J) of complex dimension m and complex structure J, K.-D . Kirchberg [19] showed that the first eigenvalue λ of the Dirac operator satisfies:

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m}\inf\limits_M S, & \text{if } m \text{ is odd,} \\ \frac{m}{4(m-1)}\inf\limits_M S, & \text{if } m \text{ is even.} \end{cases} \tag{3.2}$$

Kirchberg's estimates rely essentially on the decomposition of ΣM under the action of the Kähler form Ω. In fact, we have $\Sigma M = \oplus_{r=0}^m \Sigma_r M$, where $\Sigma_r M$ is the eigenbundle corresponding to the eigenvalue $i(2r - m)$ of Ω. The limiting manifolds of (3.2) are also characterized by the existence of spinors satisfying a certain differential equation similar to the one fulfilled by Killing spinors. More precisely, in odd complex dimension $m = 2\ell + 1$, it is proved in [21, 20, 16] that the metric is Einstein and the corresponding eigenspinor φ of λ is a Kählerian Killing spinor, *i.e.* $\varphi = \varphi_\ell + \varphi_{\ell+1} \in \Gamma(\Sigma_\ell \mathrm{M} \oplus \Sigma_{\ell+1}\mathrm{M})$ and it satisfies:

$$\begin{cases} \nabla_X\varphi_\ell = -\dfrac{\lambda}{2(m+1)}(X + iJX) \cdot \varphi_{\ell+1}, \\ \nabla_X\varphi_{\ell+1} = -\dfrac{\lambda}{2(m+1)}(X - iJX) \cdot \varphi_\ell, \end{cases} \tag{3.3}$$

for any vector field X. We point out that the existence of spinors of the form $\varphi = \varphi_{\ell'} + \varphi_{\ell'+1} \in \Gamma(\Sigma_{\ell'}\mathrm{M} \oplus \Sigma_{\ell'+1}\mathrm{M})$ satisfying (3.3), implies that m is odd and they lie in the middle, *i.e.* $l' = \frac{m-1}{2}$. If the complex dimension is even, $m = 2\ell$, the limiting manifolds are characterized by constant scalar curvature and the existence of so-called anti-holomorphic Kählerian twistor spinors $\varphi_{\ell-1} \in \Gamma(\Sigma_{\ell-1}\mathrm{M})$, *i.e.* satisfying for any vector field X: $\nabla_X\varphi_{\ell-1} = -\frac{1}{2m}(X + iJX) \cdot D\varphi_{\ell-1}$. The limiting manifolds for Kirchberg's inequalities (3.2) have been geometrically described by A. Moroianu in [28] for m odd and in [30] for m even. In [36], this result is extended to limiting manifolds of the so-called refined Kirchberg inequalities, obtained by restricting the square of the Dirac operator to the eigenbundles $\Sigma_r M$. When m is even, the limiting manifold cannot be Einstein. Thus, on compact Kähler-Einstein manifolds of even complex dimension, Kirchberg [22] improved (3.2) to the following lower bound:

$$\lambda^2 \geq \frac{m+2}{4m}S. \tag{3.4}$$

Equality is characterized by the existence of holomorphic or anti-holomorphic spinors. When m is odd, A. Moroianu extended the above mentioned result of O. Hijazi to Kähler manifolds, by showing that the Clifford multiplication between a harmonic effective form of nonzero degree and a Kählerian Killing spinor vanishes. We recall that the manifolds of complex dimension $m = 4\ell + 3$ admitting Kählerian Killing spinors are exactly the Kähler-Einstein manifolds carrying a complex contact structure (*cf.* [23], [28], [33]).

In the present paper, we extend this result of A. Moroianu to Kählerian Killing spin^c spinors (see Theorem 3.10). In this more general setting difficulties occur due to the fact that the connection on the spin^c bundle, hence its curvature, the Dirac operator and its spectrum, do not only depend on the geometry of the manifold, but also on the connection of the auxiliary line bundle associated with the spin^c structure.

Spin^c geometry became an active field of research with the advent of Seiberg-Witten theory, which has many applications to 4-dimensional geometry and topology [37, 39]. From an intrinsic point of view, almost complex, Sasaki and some classes of CR manifolds carry a canonical spin^c structure. In particular, every Kähler manifold is spin^c but not necessarily spin. For example, the complex projective space $\mathbb{C}P^m$ is spin if and only if m is odd. Moreover, from the extrinsic point of view, it seems that it is more natural to work with spin^c structures rather than spin structures [17, 34, 35]. For instance, on Kähler-Einstein manifolds of positive scalar curvature, O. Hijazi, S. Montiel and F. Urbano [17] constructed spin^c structures carrying Kählerian Killing spin^c spinors, *i.e.* spinors satisfying (3.3), where the covariant derivative is the spin^c one. In [9], M. Herzlich and A. Moroianu extended Friedrich's estimate (3.1) to compact Riemannian spin^c manifolds. This new lower bound involves only the conformal geometry of the manifold and the curvature of the auxiliary line bundle associated with the spin^c structure. The limiting case is characterized by the existence of a spin^c Killing or parallel spinor, such that the Clifford multiplication of the curvature form of the auxiliary line bundle with this spinor is proportional to it.

In this paper, we give an estimate for the eigenvalues of the spin^c Dirac operator, by restricting ourselves to compact Kähler-Einstein manifolds endowed with particular spin^c structures. More precisely, we consider (M^{2m}, g, J) a compact Kähler-Einstein manifold of positive scalar curvature S and of index $p \in \mathbb{N}^*$. We endow M with the spin^c structure whose auxiliary line bundle is a tensorial power $\mathcal{L}^q$ of the p-*th* root $\mathcal{L}$ of the canonical bundle K_M of M, where $q \in \mathbb{Z}$, $p + q \in 2\mathbb{Z}$ and $|q| \le p$. Our main result is the following:

Theorem 3.1 *Let M^{2m} be a compact Kähler-Einstein manifold of index p and positive scalar curvature S, carrying the* spin^c *structure given by $\mathcal{L}^q$ with $q + p \in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We assume that $p \ge |q|$ and the metric is normalized such that its scalar curvature equals $4m(m+1)$. Then, any eigenvalue λ of D^2 is bounded from below as follows:*

$$\lambda \ge \left(1 - \frac{q^2}{p^2}\right)(m+1)^2, \tag{3.5}$$

Equality is attained if and only if $b := \frac{q}{p} \cdot \frac{m+1}{2} + \frac{m-1}{2} \in \mathbb{N}$ and there exists a Kählerian Killing spin^c *spinor in $\Gamma(\Sigma_b M \oplus \Sigma_{b+1} M)$.*

Indeed, this is a consequence of more refined estimates for the eigenvalues of the square of the spin^c Dirac operator restricted to the eigenbundles $\Sigma_r M$ of the spinor bundle (see Theorem 3.8). The proof of this result is based on a refined Schrödinger-Lichnerowicz spin^c formula (see Lemma 3.7) written on each such eigenbundle $\Sigma_r M$, which uses the decomposition of the covariant derivative acting on spinors into its holomorphic and antiholomorphic part. This formula has already been used in literature, for instance by K.-D. Kirchberg [22]. The limiting manifolds of (3.5) are characterized by the existence of Kählerian Killing spin^c spinors in a certain subbundle $\Sigma_r M$. In particular, this gives a positive answer to the conjectured relationship between spin^c Kählerian Killing spinors and a lower bound for the eigenvalues of the spin^c Dirac operator, as stated in [17, Remark 16].
Let us mention here that the Einstein condition in Theorem 3.1 is important in order to establish the estimate (3.5), since otherwise there is no control over the estimate of the term given by the Clifford action of the curvature form of the auxiliary line bundle of the spin^c structure (see (3.25)).

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2 Preliminaries and Notation

In this section, we set the notation and briefly review some basic facts about spin^c and Kähler geometries. For more details we refer to the books [4], [7], [24] and [32].

Let (M^n, g) be an n-dimensional closed Riemannian spin^c manifold and denote by ΣM its complex spinor bundle, which has complex rank equal to $2^{[\frac{n}{2}]}$. The bundle ΣM is endowed with a Clifford multiplication denoted by "$\cdot$" and a scalar product denoted by $\langle \cdot, \cdot \rangle$. Given a spin^c structure on (M^n, g), one can check that the determinant line bundle $\det(\Sigma M)$ has a root L of index $2^{[\frac{n}{2}]-1}$. This line bundle L over M is called the auxiliary line bundle associated with the spin^c structure. The connection ∇^A on ΣM is the twisted connection of the one on the spinor bundle (induced by the Levi-Civita connection) and a fixed connection A on L. The spin^c Dirac operator D^A acting on the space of sections of ΣM is defined by the composition of the connection ∇^A with the Clifford multiplication. For simplicity, we will denote ∇^A by ∇ and D^A by D. In local coordinates:

$$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j},$$

where $\{e_j\}_{j=1,\ldots,n}$ is a local orthonormal basis of TM. D is a first-order elliptic operator and is formally self-adjoint with respect to the L^2-scalar product. A useful tool when examining the spin^c Dirac operator is the Schrödinger-Lichnerowicz formula:

$$D^2 = \nabla^*\nabla + \frac{1}{4}S + \frac{1}{2}F_A\cdot, \tag{3.6}$$

where ∇^* is the adjoint of ∇ with respect to the L^2-scalar product and F_A is the curvature (imaginary-valued) 2-form on M associated to the connection A defined on the auxiliary line bundle L, which acts on spinors by the extension of the Clifford multiplication to differential forms.

We recall that the complex volume element $\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \wedge \ldots \wedge e_n$ acts as the identity on the spinor bundle if n is odd. If n is even, $\omega_{\mathbb{C}}^2 = 1$. Thus, under the action of the complex volume element, the spinor bundle decomposes into the eigenspaces $\Sigma^{\pm}M$ corresponding to the ± 1 eigenspaces, the *positive* (resp. *negative*) spinors.

Every spin manifold has a trivial spin^c structure, by choosing the trivial line bundle with the trivial connection whose curvature F_A vanishes. Every Kähler manifold (M^{2m}, g, J) has a canonical spin^c structure induced by the complex structure J. The complexified tangent bundle decomposes into $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$, the i-eigenbundle (resp. $(-i)$-eigenbundle) of the complex linear extension of J. For any vector field X, we denote by $X^{\pm} := \frac{1}{2}(X \mp iJX)$ its component in $T_{1,0}M$, resp. $T_{0,1}M$. The spinor bundle of the canonical spin^c structure is defined by

$$\Sigma M = \Lambda^{0,*}M = \overset{m}{\underset{r=0}{\oplus}} \Lambda^r(T^*_{0,1}M),$$

and its auxiliary line bundle is $L = (K_M)^{-1} = \Lambda^m(T^*_{0,1}M)$, where $K_M = \Lambda^{m,0}M$ is the canonical bundle of M. The line bundle L has a canonical holomorphic connection, whose curvature form is given by $-i\rho$, where ρ is the Ricci form defined, for all vector fields X and Y, by $\rho(X,Y) = \mathrm{Ric}(JX,Y)$ and Ric denotes the Ricci tensor. Let us mention here the sign convention we use to define the Riemann curvature tensor, respectively the Ricci tensor: $R_{X,Y} := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ and $\mathrm{Ric}(X,Y) := \sum_{j=1}^{2m} R(e_j, X, Y, e_j)$, for all vector fields X, Y on M, where $\{e_j\}_{j=1,\ldots,2m}$ is a local orthonormal basis of the tangent bundle. Similarly, one defines the so called anti-canonical spin^c structure, whose spinor bundle is given by $\Lambda^{*,0}M = \oplus_{r=0}^m \Lambda^r(T^*_{1,0}M)$ and the auxiliary line bundle by K_M. The spinor bundle of any other spin^c structure on M can be written as:

$$\Sigma M = \Lambda^{0,*}M \otimes \mathbb{L},$$

where $\mathbb{L}^2 = K_M \otimes L$ and L is the auxiliary line bundle associated with this spin^c structure. The Kähler form Ω, defined as $\Omega(X,Y) = g(JX,Y)$, acts on ΣM via Clifford multiplication and this action is locally given by:

$$\Omega \cdot \psi = \frac{1}{2}\sum_{j=1}^{2m} e_j \cdot Je_j \cdot \psi, \tag{3.7}$$

for all $\psi \in \Gamma(\Sigma M)$, where $\{e_1, \ldots, e_{2m}\}$ is a local orthonormal basis of TM. Under this action, the spinor bundle decomposes as follows:

$$\Sigma M = \overset{m}{\underset{r=0}{\oplus}} \Sigma_r M, \tag{3.8}$$

where $\Sigma_r M$ denotes the eigenbundle to the eigenvalue $i(2r - m)$ of Ω, of complex rank $\binom{m}{k}$. It is easy to see that $\Sigma_r M \subset \Sigma^+ M$ (resp. $\Sigma_r M \subset \Sigma^- M$) if and only if r is even (resp. r is odd). Moreover, for any $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\Sigma_r M)$, we have $X^+ \cdot \varphi \in \Gamma(\Sigma_{r+1} M)$ and $X^- \cdot \varphi \in \Gamma(\Sigma_{r-1} M)$, with the convention $\Sigma_{-1}\mathrm{M} = \Sigma_{m+1}\mathrm{M} = M \times \{0\}$. Thus, for any spin^c structure, we have $\Sigma_r M = \Lambda^{0,r} M \otimes \Sigma_0 M$. Hence, $(\Sigma_0 M)^2 = K_M \otimes L$, where L is the auxiliary line bundle associated with the spin^c structure. For example, when the manifold is spin, we have $(\Sigma_0 M)^2 = K_M$ [18, 19]. For the canonical spin^c structure, since $L = (K_M)^{-1}$, it follows that $\Sigma_0 M$ is trivial. This yields the existence of parallel spinors (the constant functions) lying in $\Sigma_0 M$, *cf.* [31].

Associated to the complex structure J, one defines the following operators:

$$D^+ = \sum_{j=1}^{2m} e_j^+ \cdot \nabla_{e_j^-}, \quad D^- = \sum_{j=1}^{2m} e_j^- \cdot \nabla_{e_j^+}, \tag{3.9}$$

which satisfy the relations

$$D = D^+ + D^-, \quad (D^+)^2 = 0, \quad (D^-)^2 = 0, \quad D^+D^- + D^-D^+ = D^2. \tag{3.10}$$

When restricting the Dirac operator to $\Sigma_r\mathrm{M}$, it acts as

$$D = D^+ + D^- : \Gamma(\Sigma_r\mathrm{M}) \to \Gamma(\Sigma_{r-1}\mathrm{M} \oplus \Sigma_{r+1}\mathrm{M}).$$

Corresponding to the decomposition $\mathrm{TM} \otimes \Sigma_r\mathrm{M} \cong \Sigma_{r-1}\mathrm{M} \oplus \Sigma_{r+1}\mathrm{M} \oplus \mathrm{Ker}_r$, where Ker_r denotes the kernel of the Clifford multiplication by tangent vectors restricted to $\Sigma_r\mathrm{M}$, we have, as in the spin case (for details see *e.g.* [36, (2.7)]), the following Weitzenböck formula relating the differential operators acting on sections of $\Sigma_r\mathrm{M}$:

$$\nabla^*\nabla = \frac{1}{2(r+1)} D^- D^+ + \frac{1}{2(m-r+1)} D^+ D^- + T_r^* T_r, \tag{3.11}$$

where T_r is the so-called Kählerian twistor operator and is defined by

$$T_r\varphi := \nabla\varphi + \frac{1}{2(m-r+1)} e_j \otimes e_j^+ \cdot D^-\varphi + \frac{1}{2(r+1)} e_j \otimes e_j^- \cdot D^+\varphi.$$

This decomposition further implies the following identity for $\varphi \in \Gamma(\Sigma_r\mathrm{M})$, by the same argument as in [36, Lemma 2.5]:

$$|\nabla\varphi|^2 = \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2 + |T_r\varphi|^2. \tag{3.12}$$

Hence, we have the inequality:

$$|\nabla\varphi|^2 \geq \frac{1}{2(r+1)}|D^+\varphi|^2 + \frac{1}{2(m-r+1)}|D^-\varphi|^2. \tag{3.13}$$

Equality in (3.13) is attained if and only if $T_r\varphi = 0$, in which case φ is called a Kählerian twistor spinor. The Lichnerowicz-Schrödinger formula (3.6) yields the following:

Lemma 3.2 *Let (M^{2m}, g, J) be a compact Kähler manifold endowed with any* spin^c *structure. If φ is an eigenspinor of D^2 with eigenvalue λ, $D^2\varphi = \lambda\varphi$, and satisfies*

$$|\nabla\varphi|^2 \geq \frac{1}{j}|D\varphi|^2, \tag{3.14}$$

for some real number $j > 1$, and $(S + 2F_A)\cdot\varphi = c\,\varphi$, where c is a positive function, then

$$\lambda \geq \frac{j}{4(j-1)}\inf_M c. \tag{3.15}$$

Moreover, equality in (3.15) *holds if and only if the function c is constant and equality in* (3.14) *holds at all points of the manifold.*

Let $\{e_1, \ldots, e_{2m}\}$ be a local orthonormal basis of M^{2m}. We implicitly use the Einstein summation convention over repeated indices. We have the following formulas for contractions that hold as endomorphisms of $\Sigma_r\mathrm{M}$:

$$e_j^+ \cdot e_j^- = -2r, \quad e_j^- \cdot e_j^+ = -2(m-r), \tag{3.16}$$

$$e_j \cdot \mathrm{Ric}(e_j) = -S, \quad e_j^- \cdot \mathrm{Ric}(e_j^+) = -\frac{S}{2} - i\rho, \quad e_j^+ \cdot \mathrm{Ric}(e_j^-) = -\frac{S}{2} + i\rho. \tag{3.17}$$

The identities (3.16) follow directly from (3.7), which gives the action of the Kähler form and has $\Sigma_r\mathrm{M}$ as eigenspace to the eigenvalue $i(2r-m)$, implying that $ie_j \cdot Je_j = 2i\Omega = -2(2r-m)$, and from the fact that $e_j \cdot e_j = -2m$. The identities (3.17) are obtained from the following identities:

$$e_j \cdot \mathrm{Ric}(e_j) = e_j \wedge \mathrm{Ric}(e_j) - g(\mathrm{Ric}(e_j), e_j) = -S,$$

$$ie_j \cdot \mathrm{Ric}(Je_j) = ie_j \wedge \mathrm{Ric}(Je_j) - ig(\mathrm{Ric}(Je_j), e_j) = 2i\rho.$$

The spin^c Ricci identity, for any spinor φ and any vector field X, is given by:

$$e_i \cdot \mathcal{R}^A_{e_i,X}\varphi = \frac{1}{2}\mathrm{Ric}(X)\cdot\varphi - \frac{1}{2}(X \lrcorner F_A)\cdot\varphi, \tag{3.18}$$

where $\mathcal{R}^A$ denotes the spin^c spinorial curvature, defined with the same sign convention as above, namely $\mathcal{R}^A_{X,Y} := \nabla^A_X\nabla^A_Y - \nabla^A_Y\nabla^A_X - \nabla^A_{[X,Y]}$. For a proof of the spin^c Ricci identity we refer to [7, Section 3.1]. For any vector field X parallel at the point where the computation is done, the following commutator rules hold:

$$[\nabla_X, D] = -\frac{1}{2}\mathrm{Ric}(X)\cdot + \frac{1}{2}(X\lrcorner F_A)\cdot, \tag{3.19}$$

$$[\nabla_X, D^+] = -\frac{1}{2}\mathrm{Ric}(X^+)\cdot + \frac{1}{2}(X^+\lrcorner F_A^{1,1})\cdot + \frac{1}{2}X^-\lrcorner F_A^{0,2}\cdot, \tag{3.20}$$

$$[\nabla_X, D^-] = -\frac{1}{2}\mathrm{Ric}(X^-)\cdot + \frac{1}{2}(X^-\lrcorner F_A^{1,1})\cdot + \frac{1}{2}X^+\lrcorner F_A^{2,0}\cdot, \tag{3.21}$$

where the 2-form F_A is decomposed as $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$, into forms of type $(2,0)$, $(1,1)$, respectively $(0,2)$. The identity (3.19) is obtained from the following straightforward computation:

$$\begin{aligned}\nabla_X(D\varphi) &= \nabla_X(e_j\cdot\nabla_{e_j}\varphi) = e_j\cdot\mathcal{R}^A_{X,e_j}\varphi + e_j\cdot\nabla_{e_j}\nabla_X\varphi\\ &\overset{(3.18)}{=} -\frac{1}{2}\mathrm{Ric}(X)\cdot\varphi + \frac{1}{2}(X\lrcorner F_A)\cdot\varphi + D(\nabla_X\varphi).\end{aligned}$$

The identity (3.20) follows from the identities:

$$\begin{aligned}\nabla_{X^+}(D^+\varphi) &= \nabla_{X^+}(e_i^+\cdot\nabla_{e_i^-}\varphi) = e_i^+\cdot\mathcal{R}^A_{X^+,e_i^-}\varphi + e_i^+\cdot\nabla_{e_i^-}\nabla_{X^+}\varphi\\ &= -\frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \frac{1}{2}(X^+\lrcorner F_A^{1,1})\cdot\varphi + D^+(\nabla_{X^+}\varphi),\end{aligned}$$

$$\begin{aligned}\nabla_{X^-}(D^+\varphi) &= \nabla_{X^-}(e_i^+\cdot\nabla_{e_i^-}\varphi) = e_i^+\cdot R_{X^-,e_i^-}\varphi + e_i^+\cdot\nabla_{e_i^-}\nabla_{X^-}\varphi\\ &= \frac{1}{2}X^-\lrcorner F_A^{0,2}\cdot\varphi + D^+(\nabla_{X^-}\varphi).\end{aligned}$$

The identity (3.21) follows either by an analogous computation or by conjugating (3.20).

On a Kähler manifold (M,g,J) endowed with any spin^c structure, a spinor of the form $\varphi_r + \varphi_{r+1} \in \Gamma(\Sigma_r M \oplus \Sigma_{r+1}M)$, for some $0 \le r \le m$, is called a *Kählerian Killing* spin^c *spinor* if there exists a non-zero real constant α, such that the following equations are satisfied, for all vector fields X,

$$\begin{cases}\nabla_X\varphi_r = \alpha\, X^-\cdot\varphi_{r+1},\\ \nabla_X\varphi_{r+1} = \alpha\, X^+\cdot\varphi_r.\end{cases} \tag{3.22}$$

Kählerian Killing spinors lying in $\Gamma(\Sigma_m M \oplus \Sigma_{m+1}M) = \Gamma(\Sigma_m M)$ or in $\Gamma(\Sigma_{-1}M \oplus \Sigma_0 M) = \Gamma(\Sigma_0 M)$ are just parallel spinors. A direct computation shows that each Kählerian Killing spin^c spinor is an eigenspinor of the square of the Dirac operator. More precisely, the following equalities hold:

$$D\varphi_r = -2(r+1)\alpha\varphi_{r+1}, \quad D\varphi_{r+1} = -2(m-r)\alpha\varphi_r, \tag{3.23}$$

which further yield

$$D^2\varphi_r = 4(m-r)(r+1)\alpha^2\varphi_r, \quad D^2\varphi_{r+1} = 4(m-r)(r+1)\alpha^2\varphi_{r+1}. \tag{3.24}$$

In [17], the authors gave examples of spin^c structures on compact Kähler-Einstein manifolds of positive scalar curvature, which carry Kählerian Killing spin^c spinors lying in $\Sigma_r M \oplus \Sigma_{r+1}M$, for $r \neq \frac{m\pm 1}{2}$, in contrast to the spin case, where Kählerian Killing spinors may only exist for m odd in the middle of the decomposition (3.8). We briefly describe these spin^c structures here. If the first Chern class $c_1(K_M)$ of the canonical bundle of the Kähler M is a non-zero cohomology class, the greatest number $p \in \mathbb{N}^*$ such that

$$\frac{1}{p}c_1(K_M) \in H^2(M,\mathbb{Z}),$$

is called the *(Fano) index* of the manifold M. One can thus consider a p-th root of the canonical bundle K_M, *i.e.* a complex line bundle $\mathcal{L}$, such that $\mathcal{L}^p = K_M$. In [17], O. Hijazi, S. Montiel and F. Urbano proved the following:

Theorem 3.3 (Theorem 14, [17]) *Let M be a $2m$-dimensional Kähler-Einstein compact manifold with scalar curvature $4m(m+1)$ and index $p \in \mathbb{N}^*$. For each $0 \leq r \leq m+1$, there exists on M a* spin^c *structure with auxiliary line bundle given by $\mathcal{L}^q$, where $q = \frac{p}{m+1}(2r-m-1) \in \mathbb{Z}$, and carrying a Kählerian Killing spinor $\psi_{r-1}+\psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$,* i.e. *it satisfies the first order system*

$$\begin{cases} \nabla_X \psi_r = -X^+ \cdot \psi_{r-1}, \\ \nabla_X \psi_{r-1} = -X^- \cdot \psi_r, \end{cases}$$

for all $X \in \Gamma(TM)$.

For example, if M is the complex projective space $\mathbb{C}P^m$ of complex dimension m, then $p = m+1$ and $\mathcal{L}$ is just the tautological line bundle. We fix $0 \leq r \leq m+1$ and we endow $\mathbb{C}P^m$ with the spin^c structure whose auxiliary line bundle is given by $\mathcal{L}^q$ where $q = \frac{p}{m+1}(2r-m-1) = 2r-m-1 \in \mathbb{Z}$. For this spin^c structure, the space of Kählerian Killing spinors in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ has dimension $\binom{m+1}{r}$. A Kähler manifold carrying a complex contact structure necessarily has odd complex dimension $m = 2\ell+1$ and its index p equals $\ell+1$. We fix $0 \leq r \leq m+1$ and we endow M with the spin^c structure whose auxiliary line bundle is given by $\mathcal{L}^q$ where $q = \frac{p}{m+1}(2r-m-1) = r-\ell-1 \in \mathbb{Z}$. For this spin^c structure, the space of Kählerian Killing spinors in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ has dimension 1. In these examples, for $r = 0$ (resp. $r = m+1$), we get the canonical (resp. anticanonical) spin^c structure for which Kählerian Killing spinors are just parallel spinors.

3 Eigenvalue estimates for the spin^c Dirac operator on Kähler-Einstein Manifolds

In this section, we give a lower bound for the eigenvalues of the spin^c Dirac operator on a Kähler-Einstein manifold endowed with particular spin^c structures. More precisely, let (M^{2m}, g, J) be a compact Kähler-Einstein manifold of index $p \in \mathbb{N}^*$ and of positive scalar curvature S, endowed with the spin^c structure given by $\mathcal{L}^q$, where $\mathcal{L}$ is the p-*th* root of the canonical bundle and $q + p \in 2\mathbb{Z}$ (among all powers $\mathcal{L}^q$, only those satisfying $p + q \in 2\mathbb{Z}$ provide us a spin^c structure, *cf.* [17, Section 7]). The curvature form F_A of the induced connection A on $\mathcal{L}^q$ acts on the spinor bundle as $\frac{q}{p} i\rho$. Since (M^{2m}, g, J) is Kähler-Einstein, it follows that $\rho = \frac{S}{2m}\Omega$, where Ω is the Kähler form. Hence, for each $0 \leq r \leq m$, we have:

$$(S + 2F_A) \cdot \varphi_r = \left(1 - \frac{q}{p} \cdot \frac{2r - m}{m}\right) S\, \varphi_r, \quad \forall \varphi_r \in \Gamma(\Sigma_r \mathrm{M}). \tag{3.25}$$

Let us denote by $c_r := 1 - \frac{q}{p} \cdot \frac{2r-m}{m}$ and

$$a_1 : \{0, \dots, m\} \to \mathbb{R}, \qquad a_1(r) := \frac{r+1}{2r+1} c_r,$$
$$a_2 : \{0, \dots, m\} \to \mathbb{R}, \qquad a_2(r) := \frac{m-r+1}{2m-2r+1} c_r.$$

With the above notation, the following result holds:

Proposition 3.4 *Each eigenvalue λ_r of D^2 restricted to $\Sigma_r \mathrm{M}$ satisfies the inequality:*

$$\lambda_r \geq \max\Big(\min\Big(a_1(r), a_1(r-1)\Big), \min\Big(a_2(r), a_2(r+1)\Big)\Big) \cdot \frac{S}{2}. \tag{3.26}$$

Moreover, the equality case is characterized as follows:

a) $D^2 \varphi_r = a_1(r) \frac{S}{2} \varphi_r \Longleftrightarrow T_r \varphi_r = 0, D^- \varphi_r = 0.$

b) $D^2 \varphi_r = a_1(r-1) \frac{S}{2} \varphi_r \Longleftrightarrow T_{r-1}(D^- \varphi_r) = 0.$

c) $D^2 \varphi_r = a_2(r) \frac{S}{2} \varphi_r \Longleftrightarrow T_r \varphi_r = 0, D^+ \varphi_r = 0.$

d) $D^2 \varphi_r = a_2(r+1) \frac{S}{2} \varphi_r \Longleftrightarrow T_{r+1}(D^+ \varphi_r) = 0.$

Proof. For $0 \leq r \leq m$ we have: $(S + 2F_A) \cdot \varphi_r = c_r S\, \varphi_r$, $\forall \varphi_r \in \Gamma(\Sigma_r \mathrm{M})$. Let $r \in \{0, \dots, m\}$ be fixed, λ_r be an eigenvalue of $D^2|_{\Sigma_r M}$ and $\varphi \in \Gamma(\Sigma_r \mathrm{M})$ be an eigenspinor: $D^2 \varphi = \lambda_r \varphi$. We distinguish two cases.

i) If $D^- \varphi = 0$, then $|D\varphi|^2 = |D^+ \varphi|^2$ and (3.13) implies:

$$|\nabla \varphi|^2 \geq \frac{1}{2(r+1)} |D^+ \varphi|^2 = \frac{1}{2(r+1)} |D\varphi|^2.$$

By Lemma 3.2, it follows that

$$\lambda_r \geq \frac{r+1}{2(2r+1)} c_r S.$$

ii) If $D^-\varphi \neq 0$, then we consider $\varphi^- := D^-\varphi$, which satisfies $D^2\varphi^- = \lambda_r\varphi^-$ and $D^-\varphi^- = 0$, so in particular $|D\varphi^-|^2 = |D^+\varphi^-|^2$. We now apply the argument in i) to $\varphi^- \in \Gamma(\Sigma_{r-1}\mathrm{M})$. By (3.13), it follows that

$$|\nabla\varphi^-|^2 \geq \frac{1}{2r}|D^+\varphi^-|^2 = \frac{1}{2r}|D\varphi^-|^2.$$

Applying again Lemma 3.2, we obtain $\lambda_r \geq \frac{r}{2(2r-1)}c_{r-1}S$.

Hence, we have showed that $\lambda_r \geq \min\Big(a_1(r), a_1(r-1)\Big)\frac{S}{2}$. The same argument applied to the cases when $D^+\varphi = 0$ and $D^+\varphi \neq 0$ proves the inequality $\lambda_r \geq \min\Big(a_2(r), a_2(r+1)\Big)\frac{S}{2}$. Altogether we obtain the estimate in Proposition 3.4. The characterization of the equality cases is a direct consequence of Lemma 3.2, identity (3.12) and the description of the limiting case of inequality (3.13). □

Remark 3.5 *The inequality* (3.26) *can be expressed more explicitly, by determining the maximum according to several possible cases. However, since in the sequel we will refine this eigenvalue estimate, we are only interested in the characterization of the limiting cases, which will be used later in the proof of the equality case of the estimate (3.5).*

In order to refine the estimate (3.26), we start by the following two lemmas.

Lemma 3.6 *Let (M^{2m}, g, J) be a compact Kähler-Einstein manifold of index p and of positive scalar curvature S, endowed with a* spin^c *structure given by $\mathcal{L}^q$, where $q+p \in 2\mathbb{Z}$. For any spinor field φ and any vector field X, the* spin^c *Ricci identity is given by:*

$$e_j \cdot \mathcal{R}^A_{e_j,X}\varphi = \frac{1}{2}\mathrm{Ric}(X)\cdot\varphi - \frac{S}{4m}\frac{q}{p}(X\lrcorner i\Omega)\cdot\varphi, \tag{3.27}$$

and it can be refined as follows:

$$e_j^- \cdot \mathcal{R}^A_{e_j^+,X^-}\varphi = \frac{1}{2}\mathrm{Ric}(X^-)\cdot\varphi - \frac{S}{4m}\frac{q}{p}X^-\cdot\varphi, \tag{3.28}$$

$$e_j^+ \cdot \mathcal{R}^A_{e_j^-,X^+}\varphi = \frac{1}{2}\mathrm{Ric}(X^+)\cdot\varphi + \frac{S}{4m}\frac{q}{p}X^+\cdot\varphi. \tag{3.29}$$

Proof. Since the curvature form F_A of the spin^c structure acts on the spinor bundle as $\frac{q}{p}i\rho = \frac{q}{p}\frac{S}{2m}i\Omega$, (3.27) follows directly from the Ricci identity (3.18). The refined identities (3.28) and (3.29) follow by replacing X in (3.27) with X^-, respectively X^+, which is possible since both sides of the identity are complex linear in X, and by taking into account that when decomposing $e_j = e_j^+ + e_j^-$, the following identities (and their analogue for X^+) hold: $e_j \cdot \mathcal{R}^A_{e_j^-,X^-} = 0$ and $e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-} = 0$. These last two identities are a consequence of the J-invariance of the curvature tensor, *i.e.* $\mathcal{R}^A_{JX,JY} = \mathcal{R}^A_{X,Y}$, for all vector fields X, Y, as this implies $\mathcal{R}^A_{e_j^-,X^-} = \mathcal{R}^A_{Je_j^-,JX^-} = (-i)^2\mathcal{R}^A_{e_j^-,X^-}$ and also $e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-} = Je_j^+ \cdot \mathcal{R}^A_{Je_j^+,X^-} = i^2 e_j^+ \cdot \mathcal{R}^A_{e_j^+,X^-}$, so they both vanish. In order to obtain the second term on the right hand side of (3.28) and (3.29), we use the following identities of endomorphisms of the spinor bundle: $X^- \lrcorner i\Omega = X^-$ and $X^+ \lrcorner i\Omega = -X^+$. $\square$

Lemma 3.7 *Under the same assumptions as in Lemma 3.6, the refined Schrödinger-Lichnerowicz formula for* spin^c *Kähler manifolds for the action on each eigenbundle* $\Sigma_r\mathrm{M}$ *is given by*

$$2\nabla^{1,0^*}\nabla^{1,0} = D^2 - \frac{S}{4} - \frac{i}{2}\rho - \frac{m-r}{2m}\frac{q}{p}S, \tag{3.30}$$

$$2\nabla^{0,1^*}\nabla^{0,1} = D^2 - \frac{S}{4} + \frac{i}{2}\rho + \frac{r}{2m}\frac{q}{p}S, \tag{3.31}$$

where $\nabla^{1,0}$ *(resp.* $\nabla^{0,1}$*) is the holomorphic (resp. antiholomorphic) part of* ∇*,* i.e. *the projections of* ∇ *onto the following two components:*

$$\nabla\colon \Gamma(\Sigma_r\mathrm{M}) \to \Gamma(\Lambda^{1,0}M \otimes \Sigma_r\mathrm{M}) \oplus \Gamma(\Lambda^{0,1}M \otimes \Sigma_r\mathrm{M}).$$

They are locally defined, for all vector fields X, by

$$\nabla^{1,0}_X = g(X, e_i^-)\nabla_{e_i^+} = \nabla_{X^+} \quad \textit{and} \quad \nabla^{0,1}_X = g(X, e_i^+)\nabla_{e_i^-} = \nabla_{X^-},$$

where $\{e_1, \ldots, e_{2m}\}$ *is a local orthonormal basis of* TM.

Proof. Let $\{e_1, \ldots, e_{2m}\}$ be a local orthonormal basis of TM (identified with $\Lambda^1 M$ via the metric g), parallel at the point where the computation is made. We recall that the formal adjoints $\nabla^{1,0^*}$ and $\nabla^{1,0^*}$ are given by the following formulas (for a proof, see *e.g.* [32, Lemma 20.1]):

$$\nabla^{1,0^*}\colon \Gamma(\Lambda^{1,0}M \otimes \Sigma_r\mathrm{M}) \longrightarrow \Gamma(\Sigma_r\mathrm{M}), \quad \nabla^{1,0^*}(\alpha \otimes \varphi) = (\delta\alpha)\varphi - \nabla_\alpha\varphi,$$

$$\nabla^{0,1^*}\colon \Gamma(\Lambda^{0,1}M \otimes \Sigma_r\mathrm{M}) \longrightarrow \Gamma(\Sigma_r\mathrm{M}), \quad \nabla^{0,1^*}(\alpha \otimes \varphi) = (\delta\alpha)\varphi - \nabla_\alpha\varphi.$$

We thus obtain for the corresponding Laplacians:

$$\nabla^{1,0^*}\nabla^{1,0}\varphi = \nabla^{1,0^*}(e_j^- \otimes \nabla_{e_j^+}\varphi) = -\nabla_{e_j^-}\nabla_{e_j^+}, \tag{3.32}$$

since $\delta e_j^- = 0$, as the basis is parallel at the given point, and $g(\cdot, e_j^-) \in \Lambda^{1,0}M$. Analogously, or by conjugation, we have $\nabla^{0,1^*}\nabla^{0,1}\varphi = -\nabla_{e_j^+}\nabla_{e_j^-}$. We now prove (3.30). By a similar computation, one obtains (3.31).

$$\begin{aligned}
2\nabla^{1,0^*}\nabla^{1,0} &\overset{(3.32)}{=} -2g(e_i,e_j)\nabla_{e_i^-}\nabla_{e_j^+} = (e_i\cdot e_j + e_j\cdot e_i)\cdot\nabla_{e_i^-}\nabla_{e_j^+}\\
&= D^+D^- + e_j\cdot e_i\cdot(\nabla_{e_j^+}\nabla_{e_i^-} - R_{e_j^+,e_i^-})\\
&\overset{(3.29)}{=} D^+D^- + D^-D^+ + e_j^-\cdot e_i^+\cdot R_{e_i^-,e_j^+}\\
&= D^2 + e_j^-\cdot\left(\frac{1}{2}\mathrm{Ric}(e_j^+) + \frac{S}{4m}\frac{q}{p}e_j^+\right)\\
&\overset{(3.16),(3.17)}{=} D^2 - \frac{1}{2}\left(\frac{S}{2}+i\rho\right) - \frac{m-r}{2m}\frac{q}{p}S.
\end{aligned}$$

□

Theorem 3.8 *Let (M^{2m},g,J) be a compact Kähler-Einstein manifold of index p and positive scalar curvature S, carrying the* spin^c *structure given by $\mathcal{L}^q$ with $q+p\in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We assume that $p\geq |q|$. Then, for each $r\in\{0,\dots,m\}$, any eigenvalue λ_r of $D^2|_{\Gamma(\Sigma_r M)}$ satisfies the inequality:*

$$\lambda_r \geq e(r)\frac{S}{2}, \tag{3.33}$$

where

$$e\colon [0,m]\to\mathbb{R},\ e(x) = \begin{cases} e_1(x) = \frac{m-x}{m}\left(1+\frac{q}{p}\right), & \text{if } x\leq \left(1+\frac{q}{p}\right)\frac{m}{2},\\ e_2(x) = \frac{x}{m}\left(1-\frac{q}{p}\right), & \text{if } x\geq \left(1+\frac{q}{p}\right)\frac{m}{2}.\end{cases}$$

Moreover, equality is attained if and only if the corresponding eigenspinor $\varphi_r\in\Gamma(\Sigma_r M)$ is an antiholomorphic spinor: $\nabla^{1,0}\varphi_r = 0$, if $r\leq\left(1+\frac{q}{p}\right)\frac{m}{2}$, respectively a holomorphic spinor: $\nabla^{0,1}\varphi_r = 0$, if $r\geq\left(1+\frac{q}{p}\right)\frac{m}{2}$.

Proof. First we notice that our assumption $|q|\leq p$ implies that the lower bound in (3.33) is non-negative and that $0\leq\left(1+\frac{q}{p}\right)\frac{m}{2}\leq m$. The formulas (3.30) and (3.31) applied to φ_r yield, after taking the scalar product with φ_r and integrating over M, the following inequalities:

$$\lambda_r \geq \frac{m-r}{m}\left(1+\frac{q}{p}\right)\frac{S}{2},$$
$$\lambda_r \geq \frac{r}{m}\left(1-\frac{q}{p}\right)\frac{S}{2},$$

and equality is attained if and only if the corresponding eigenspinor φ_r satisfies $\nabla^{1,0}\varphi_r = 0$, resp. $\nabla^{0,1}\varphi_r = 0$. Hence, for any $0\leq r\leq m$ we obtain the following lower bound:

$$\lambda_r \geq \max\left(\frac{m-r}{m}\left(1+\frac{q}{p}\right), \frac{r}{m}\left(1-\frac{q}{p}\right)\right) = e(r)\frac{S}{2}.$$

□

Remark 3.1 *Let us denote $\frac{q}{p} \cdot \frac{m+1}{2} + \frac{m-1}{2}$ by b. Comparing the estimate given by Theorem 3.8 with the estimate from Proposition 3.4, we obtain for $r \leq b$:*

$$e(r) - a_1(r) = \frac{(m+1)\frac{q}{p} + m - 1 - 2r}{m(2r+1)} = -\frac{2(r-b)}{m(2r+1)}.$$

Hence, for $r \leq b$, we have $e(r) \geq a_1(r)$ and $e(r) = a_1(r)$ iff $r = b \in \mathbb{N}$.
Similarly, for $r \geq b+1$, we compute:

$$e(r) - a_2(r) = \frac{2(m-r)(r-b-1)}{m(2m-2r+1)}.$$

Hence, for $r \geq b+1$, we have $e(r) \geq a_2(r)$ and $e(r) = a_2(r)$ iff $r = b+1 \in \mathbb{N}$.

Theorem 3.8 implies the global lower bound for the eigenvalues of the spin^c Dirac operator acting on the whole spinor bundle in Theorem 3.1. We are now ready to prove this result.

Proof of Theorem 3.1 Since the lower bound established in Theorem 3.8 decreases on $(0, \left(1+\frac{q}{p}\right)\frac{m}{2})$ and increases on $(\left(1+\frac{q}{p}\right)\frac{m}{2}, m)$, we obtain the following global estimate:

$$\lambda \geq e\left(\left(1+\frac{q}{p}\right)\frac{m}{2}\right) = \frac{1}{2}\left(1-\frac{q^2}{p^2}\right)\frac{S}{2}.$$

However, this estimate is not sharp. Otherwise, this would imply that $\left(1+\frac{q}{p}\right)\frac{m}{2} \in \mathbb{N}$ and the limiting eigenspinor would be, according to the characterization of the equality case in Theorem 3.8, both holomorphic and antiholomorphic, hence parallel and, in particular, harmonic. This fact together with the Lichnerowicz-Schrödinger formula (3.6) and the fact that the scalar curvature is positive leads to a contradiction.

We now assume that there exists an $r \in \mathbb{N}$, such that $b < r < \left(1+\frac{q}{p}\right)\frac{m}{2}$ and the equality in (3.33) is attained. We obtain a contradiction as follows. Let φ_r be the corresponding eigenspinor: $D^2\varphi_r = e_1(r)\frac{S}{2}\varphi_r$ and $\nabla^{1,0}\varphi_r = 0$. Then $D^+\varphi_r \in \Sigma_{r+1}M$ is also an eigenspinor of D^2 to the eigenvalue $e_1(r)\frac{S}{2}$ (note that $D^+\varphi_r \neq 0$, otherwise φ_r would be a harmonic spinor and we could conclude as above). However, for all $r > b$, the strict inequality $e_2(r+1) > e_1(r)$ holds. Since $r+1 > \left(1+\frac{q}{p}\right)\frac{m}{2}$, this contradicts the estimate (3.33). The same argument as above shows that there exists no $r \in \mathbb{N}$, such that $\left(1+\frac{q}{p}\right)\frac{m}{2} < r < b+1$ and the equality in (3.33) is attained. Hence, we obtain the following global estimate:

$$\lambda \geq e_1(b)\frac{S}{2} = e_2(b+1)\frac{S}{2} = \frac{m+1}{2m}\left(1-\frac{q^2}{p^2}\right)\frac{S}{2} = \left(1-\frac{q^2}{p^2}\right)(m+1)^2.$$

According to Theorem 3.8, the equality is attained if and only if $b \in \mathbb{N}$ and the corresponding eigenspinors $\varphi_b \in \Gamma(\Sigma_b M)$ and $\varphi_{b+1} \in \Gamma(\Sigma_{b+1}M)$ to the eigenvalue $\left(1-\frac{q^2}{p^2}\right)(m+1)^2$ are antiholomorphic resp. holomorphic spinors: $\nabla^{1,0}\varphi_b = 0$, $\nabla^{0,1}\varphi_{b+1} = 0$. In particular, this implies $D^-\varphi_b = 0$ and $D^+\varphi_{b+1} = 0$. By Remark 3.1, we have: $e_1(b) = a_1(b)$ and $e_2(b+1) = a_2(b+1)$. Hence, the characterization of the equality case in Proposition 3.4 yields $T_b\varphi_b = 0$ and $T_{b+1}\varphi_{b+1} = 0$, which further imply:

$$\nabla_X\varphi_b = -\frac{1}{2(b+1)}X^- \cdot D^+\varphi_b = -\frac{1}{2(b+1)}X^- \cdot D\varphi_b, \tag{3.34}$$

$$\nabla_X\varphi_{b+1} = -\frac{1}{2(m-b)}X^+ \cdot D^-\varphi_{b+1} = -\frac{1}{2(m-b)}X^+ \cdot D\varphi_{b+1}. \tag{3.35}$$

We now show that the spinors $\varphi_b + \frac{1}{(m+1)\left(1+\frac{q}{p}\right)}D\varphi_b \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1}M)$ and $\varphi_{b+1} + \frac{1}{(m+1)\left(1-\frac{q}{p}\right)}D\varphi_{b+1} \in \Gamma(\Sigma_{b+1}M \oplus \Sigma_b M)$ are Kählerian Killing spin^c spinors. Note that for $q = 0$ (corresponding to the spin case), it follows that $\varphi_b + \frac{1}{m+1}D\varphi_b, \varphi_{b+1} + \frac{1}{m+1}D\varphi_{b+1} \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1}M)$ are eigenspinors of the Dirac operator corresponding to the smallest possible eigenvalue $m+1$, *i.e.* Kählerian Killing spinors. From (3.34) it follows:

$$\nabla_X\varphi_b = -X^- \cdot \frac{1}{(m+1)\left(1+\frac{q}{p}\right)}D\varphi_b \tag{3.36}$$

Applying (3.20) to φ_b in this case for $\mathrm{Ric} = \frac{S}{2m}g = 2(m+1)g$ and $F_A = \frac{q}{p}\frac{S}{2m}i\Omega = 2(m+1)\frac{q}{p}i\Omega$, we get:

$$\nabla_X(D^+\varphi_b) = -(m+1)\left(1+\frac{q}{p}\right)X^+ \cdot \varphi_b. \tag{3.37}$$

According to the definition equation (3.22) of a Kählerian Killing spin^c spinor, equations (3.36) and (3.37) imply that the spinor $\varphi_b + \frac{1}{(m+1)\left(1+\frac{q}{p}\right)}D\varphi_b \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1}M)$ is a Kählerian Killing spin^c spinor. A similar computation yields that $\varphi_{b+1} + \frac{1}{(m+1)\left(1-\frac{q}{p}\right)}D\varphi_{b+1}$ is a Kählerian Killing spin^c spinor. Conversely, if $\varphi_b + \varphi_{b+1} \in \Gamma(\Sigma_b M \oplus \Sigma_{b+1}M)$ is a Kählerian Killing spin^c spinor, then according to (3.24), φ_b and φ_{b+1} are eigenspinors of D^2 to the eigenvalue $4(m-b)(b+1) = \left(1-\frac{q^2}{p^2}\right)(m+1)^2$. This concludes the proof. $\square$

Remark 3.2 *If $q = 0$, which corresponds to the spin case, the assumption $p \geq |q| = 0$ is trivial and we recover from Theorem 3.8 and Theorem 3.1 Kirchberg's estimates on Kähler-Einstein spin manifolds: the lower bound* (3.2) *for m odd, namely $\lambda^2 \geq \frac{m+1}{4m}S = e(\frac{m+1}{2})\frac{S}{2}$, and the lower bound* (3.4) *for m even, namely $\lambda^2 \geq \frac{m+2}{4m}S = e(\frac{m}{2}+1)\frac{S}{2}$. If $q = -p$ (resp. $q = p$), which corresponds to the canonical (resp. anti-canonical)* spin^c *structure, the lower bound in Theorem 3.1 equals* 0 *and is attained by the parallel spinors in $\Sigma_0 M$ (resp. $\Sigma_m M$),* cf. *[31].*

Remark 3.3 *If $q = 0$, which corresponds to the spin case, the assumption $p \geq |q| = 0$ is trivial and we recover from Theorem 3.8 and Theorem 3.1 Kirchberg's estimates on Kähler-Einstein spin manifolds: the lower bound* (3.2) *for m odd, namely $\lambda^2 \geq \frac{m+1}{4m}S = e(\frac{m+1}{2})\frac{S}{2}$, and the lower bound* (3.4) *for m even, namely $\lambda^2 \geq \frac{m+2}{4m}S = e(\frac{m}{2}+1)\frac{S}{2}$. In the latter case, when m is even, the equality in* (3.5) *cannot be attained, as $b = \frac{m}{2} - \frac{1}{2} \notin \mathbb{N}$. Also for $r = \frac{m}{2}$ the inequality* (3.33) *is strict, since otherwise it would imply, according to the characterization of the equality case in Theorem 3.8, that the corresponding eigenspinor $\varphi \in \Sigma_{\frac{m}{2}}\mathrm{M}$ is parallel, in contradiction to the positivity of the scalar curvature. Note that the same argument as in the proof of Theorem 3.1 shows that there cannot exist an eigenspinor $\varphi \in \Sigma_{\frac{m}{2}}\mathrm{M}$ of D^2 to an eigenvalue strictly smaller than the lowest bound for $r = \frac{m}{2} \pm 1$, since otherwise $D^+\varphi$ and $D^-\varphi$ would either be eigenspinors or would vanish, leading in both cases to a contradiction. Hence, from the estimate* (3.33) *and the fact that the function e_1 decreases on $(0, \left(1+\frac{q}{p}\right)\frac{m}{2})$ and e_2 increases on $(\left(1+\frac{q}{p}\right)\frac{m}{2}, m)$, it follows that the lowest possible bound for λ^2 in this case is given by $e_1(\frac{m}{2}-1)S = e_2(\frac{m}{2}+1)S = \frac{m+2}{4m}$. If $q = -p$ (resp. $q = p$), which corresponds to the canonical (resp. anti-canonical)* spin^c *structure, the lower bound in Theorem 3.1 equals* 0 *and is attained by the parallel spinors in* $\Sigma_0\mathrm{M}$ *(resp.* $\Sigma_m\mathrm{M}$*),* cf. *[31].*

4 Harmonic forms on limiting Kähler-Einstein manifolds

In this section we give an application for the eigenvalue estimate of the spin^c Dirac operator established in Theorem 3.1. Namely, we extend to spin^c spinors the result of A. Moroianu [29] stating that the Clifford multiplication between a harmonic effective form of nonzero degree and a Kählerian Killing spinor vanishes. As above, M denotes a $2m$-dimensional Kähler-Einstein compact manifold of index p and normalized scalar curvature $4m(m+1)$, which carries the spin^c structure given by $\mathcal{L}^q$ with $q + p \in 2\mathbb{Z}$, where $\mathcal{L}^p = K_M$. We call M a *limiting manifold* if equality in (3.5) is achieved on M, which is by Theorem 3.1 equivalent to the existence of a Kählerian Killing spin^c spinor in $\Sigma_r M \oplus \Sigma_{r+1} M$ for $r = \frac{q}{p} \cdot \frac{m+1}{2} + \frac{m-1}{2} \in \mathbb{N}$. Let $\psi = \psi_{r-1} + \psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ be such a spinor, *i.e.* $\Omega \cdot \psi_{r-1} = i(2r-2-m)\psi_{r-1}$, $\Omega \cdot \psi_r = i(2r-m)\psi_r$ and the following equations are satisfied:

$$\begin{cases} \nabla_{X^+}\psi_r = -X^+ \cdot \psi_{r-1}, \\ \nabla_{X^-}\psi_{r-1} = -X^- \cdot \psi_r. \end{cases}$$

By (3.23), we have:

$$D\psi_r = 2(m-r+1)\psi_{r-1}, \quad D\psi_{r-1} = 2r\psi_r.$$

Recall that a form ω on a Kähler manifold is called *effective* if $\Lambda\omega = 0$, where Λ is the adjoint of the operator $L\colon \Lambda^* M \longrightarrow \Lambda^{*+2}M$, $L(\omega) := \omega \wedge \Omega$. More precisely, Λ is given by the formula: $\Lambda = -2\sum_{j=1}^{2m} e_j^+ \lrcorner e_j^- \lrcorner$. Moreover, one can check that

$$(\Lambda L - L\Lambda)\omega = (m-t)\omega, \; \forall \omega \in \Lambda^t M. \tag{3.38}$$

Lemma 3.9 *Let $\psi = \psi_{r-1} + \psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ be a Kählerian Killing* spin^c *spinor and ω a harmonic effective form of type (k, k'). Then, we have*

$$D(\omega \cdot \psi_r) = 2(-1)^{k+k'}(m - r + 1 - k')\omega \cdot \psi_{r-1} \tag{3.39}$$

$$D(\omega \cdot \psi_{r-1}) = 2(-1)^{k+k'}(r - k)\omega \cdot \psi_r. \tag{3.40}$$

Proof. The following general formula holds for any form ω of degree $\deg(\omega)$ and any spinor φ:

$$D(\omega \cdot \varphi) = (d\omega + \delta\omega) \cdot \varphi + (-1)^{\deg(\omega)}\omega \cdot D\varphi - 2\sum_{j=1}^{2m}(e_j \lrcorner \omega) \cdot \nabla_{e_j}\varphi.$$

Applying this formula to an effective harmonic form ω of type (k, k') and to the components of the Kählerian Killing spin^c spinor ψ, we obtain:

$$\begin{aligned} D(\omega \cdot \psi_r) &= (-1)^{k+k'}\omega \cdot D\psi_{r-1} - 2\sum_{j=1}^{2m}(e_j \lrcorner \omega) \cdot \nabla_{e_j}\psi_r \\ &= (-1)^{k+k'}2(m - r + 1)\omega \cdot \psi_{r-1} + 2\sum_{j=1}^{2m}(e_j^- \lrcorner \omega) \cdot e_j^+ \cdot \psi_{r-1} \\ &= 2(-1)^{k+k'}[(m - r + 1)\omega \cdot \psi_{r-1} + (\sum_{j=1}^{2m} e_j^+ \wedge (e_j^- \lrcorner \omega)) \cdot \psi_{r-1}] \end{aligned}$$

Since ω is effective, we have for any spinor φ that

$$(e_j^- \lrcorner \omega) \cdot e_j^+ \cdot \varphi = (-1)^{k+k'-1}\left(e_j^+ \wedge (e_j^- \lrcorner \omega) + e_j^+ \lrcorner e_j^- \lrcorner \omega\right) \cdot \varphi.$$

Thus, we conclude $D(\omega \cdot \psi_r) = 2(-1)^{k+k'}(m - r + 1 - k')\omega \cdot \psi_{r-1}$. Analogously we obtain $D(\omega \cdot \psi_{r-1}) = 2(-1)^{k+k'}(r - k)\omega \cdot \psi_r$. □

Now, we are able to state the main result of this section, which extends the result of A. Moroianu mentioned in the introduction to the spin^c setting:

Theorem 3.10 *On a compact Kähler-Einstein limiting manifold, the Clifford multiplication of a harmonic effective form of nonzero degree with the corresponding Kählerian Killing* spin^c *spinor vanishes.*

Proof. Equations (3.39) and (3.40) imply that

$$D^2(\omega \cdot \psi) = 4(r - k)(m - r + 1 - k')\omega \cdot \psi.$$

Note that for all values of $k, k' \in \{0, \dots, m\}$ and $r \in \{0, \dots, m + 1\}$, either $4(r - k)(m - r + 1 - k') \leq 0$, or $4(r - k)(m - r + 1 - k') < 4r(m - r + 1)$, which for $r = b + 1$ is exactly the lower bound obtained in Theorem 3.1 for the eigenvalues of D^2. This shows that $\omega \cdot \psi = 0$. $\square$

Kähler-Einstein manifolds carrying a complex contact structure are examples of odd dimensional Kähler manifolds with Kählerian Killing spin^c spinors in $\Sigma_{r-1}M \oplus \Sigma_r M$ for the spin^c structure (described in the introduction) whose auxiliary line bundle is given by $\mathcal{L}^q$ and $q = r - \ell - 1$, where $m = 2\ell + 1$. Thus, the result of A. Moroianu is obtained as a special case of Theorem 3.10.

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Chapter 4

The holonomy of locally conformally Kähler metrics

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Abstract. A locally conformally Kähler manifold is a complex manifold (M, J) together with a Hermitian metric g which is conformal to a Kähler metric in the neighbourhood of each point. In this paper we obtain three classification results in locally conformally Kähler geometry. The first one is the classification of conformal classes on compact manifolds containing two non-homothetic Kähler metrics. The second one is the classification of compact Einstein locally conformally Kähler manifolds. The third result is the classification of the possible (restricted) Riemannian holonomy groups of compact locally conformally Kähler manifolds. We show that every locally (but not globally) conformally Kähler compact manifold of dimension $2n$ has holonomy $\mathrm{SO}(2n)$, unless it is Vaisman, in which case it has restricted holonomy $\mathrm{SO}(2n-1)$. We also show that the restricted holonomy of a proper globally conformally Kähler compact manifold of dimension $2n$ is either $\mathrm{SO}(2n)$, or $\mathrm{SO}(2n-1)$, or $\mathrm{U}(n)$, and we give the complete description of the possible manifolds in the last two cases.

Keywords lcK manifold, holonomy, irreducibility, Kähler structure, Einstein lcK metric, conformally Kähler.

1 Introduction

It is well-known that on a compact complex manifold, any conformal class admits at most one Kähler metric, up to a positive constant. The situation might change if the complex structure is not fixed. One may thus naturally ask the following question: are there any compact manifolds which admit two non-homothetic metrics in the same conformal class, which are both Kähler (then necessarily with respect to non-conjugate complex structures)? One of the aims of the present paper is to answer this question by describing all such manifolds. This problem can be interpreted in terms of conformally Kähler metrics in real dimension $2n$ with Riemannian holonomy contained in the unitary

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group $\mathrm{U}(n)$. More generally, we want to classify *locally* conformally Kähler metrics on compact manifolds which are Einstein or have non-generic holonomy.

Recall that a Hermitian manifold (M, g, J) of complex dimension $n \geq 2$ is called locally conformally Kähler (lcK) if around every point in M the metric g can be conformally rescaled to a Kähler metric. If $\Omega := g(J\cdot, \cdot)$ denotes the fundamental 2-form, the above condition is equivalent to the existence of a closed 1-form θ, called the Lee form (which is up to a constant equal to the logarithmic differential of the local conformal factors), such that

$$\mathrm{d}\Omega = 2\theta \wedge \Omega.$$

If the Lee form θ vanishes, the structure (g, J) is simply Kähler. If the Lee form does not vanish identically, the lcK structure is called *proper*. When θ is exact, there exists a Kähler metric in the conformal class of g, and the manifold is called *globally conformally Kähler* (gcK). If θ is not exact, then (M, g, J) it is called *strictly lcK*. A particular class of proper lcK manifolds is the one consisting of manifolds whose Lee form is parallel with respect to the Levi-Civita connexion of the metric, called Vaisman manifolds. A Vaisman manifold is always strictly lcK since the Lee form, being harmonic, cannot be exact.

In this paper we study three apparently independent – but actually interrelated – classification problems:

P1. The classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with g Einstein.

P2. The classification of compact conformal manifolds (M^{2n}, c) whose conformal class c contains two non-homothetic Kähler metrics.

P3. The classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with reduced (*i.e.* non-generic) holonomy: $\mathrm{Hol}(M, g) \subsetneq \mathrm{SO}(2n)$.

It turns out that P1 and P2 are important steps (but also interesting for their own sake) towards the solution of P3.

We are able to solve each of these problems completely. The solutions are provided by Theorem 4.1, Theorem 4.2 and Theorem 4.3 below. We now explain briefly these results and describe the methods used to prove them.

In complex dimension 2, C. LeBrun [16] showed that if a compact complex surface admits a Hermitian Einstein non-Kähler metric, then the metric is gcK and the complex surface is obtained from $\mathbb{C}P^2$ by blowing up one, two or three points in general position. When the complex dimension is greater than 2, A. Derdzinski and G. Maschler, [11], have obtained the local classification of conformally-Einstein Kähler metrics, and showed that in the compact case the only Kähler metrics which are conformal (but not homothetic) to an Einstein metric are those constructed by L. Bérard-Bergery in [5]. By changing the point of view, this can be interpreted as the classification of compact (proper) globally conformally Kähler manifolds (M^{2n}, g, J, θ) with g Einstein. In order to solve P1, it remains to understand the strictly lcK case.

Since every strictly lcK manifold has infinite fundamental group, Myers' theorem shows that the scalar curvature of any compact Einstein strictly lcK manifold is non-positive. In Theorem 4.9 below we show, using Weitzenböck-type arguments, that the

Lee form of every compact Einstein lcK manifold with non-positive scalar curvature vanishes. This gives our first result:

Theorem 4.1 *If (g, J, θ) is an Einstein proper lcK structure on a compact manifold M^{2n}, then the Lee form is exact ($\theta = \mathrm{d}\varphi$), and the scalar curvature of g is positive. For $n = 2$ the complex surface (M, J) is obtained from $\mathbb{C}P^2$ by blowing up one, two or three points in general position. For $n \geq 3$, the Kähler manifold $(M, e^{-2\varphi}g, J)$ is one of the examples of conformally-Einstein Kähler manifolds constructed by Bérard-Bergery in [5].*

The solution of Problem P2 relies on Theorem 4.17 below, where we show that if (M^{2n}, g, J, θ) is a compact proper lcK manifold whose metric g is Kähler with respect to some complex structure I, then I commutes with J and the Lee form is exact: $\theta = \mathrm{d}\varphi$. In particular (g, I) and $(e^{-2\varphi}g, J)$ are Kähler structures on M, *i.e.* the conformal structure $[g]$ is ambikähler, according to the terminology introduced in dimension 4 by V. Apostolov, D. Calderbank and P. Gauduchon in [2].

Examples of ambikähler structures in every complex dimension $n \geq 2$ can be obtained on the total spaces of some S^2-bundles over compact Hodge manifolds, by an Ansatz which is reminiscent of Calabi's construction [8]. This construction is described in Example 4.19 below.

Conversely, we have the following result, which answers Problem P2:

Theorem 4.2 *Assume that a conformal class on a compact manifold M of real dimension $2n \geq 4$ contains two non-homothetic Kähler metrics g_+ and g_-, that is, there exist complex structures J_+ and J_- and a non-constant function φ such that (g_+, J_+) and $(g_- := e^{-2\varphi}g_+, J_-)$ are Kähler structures. Then J_+ and J_- commute, so that M is ambikähler. Moreover, for $n \geq 3$, there exists a compact Kähler manifold (N, h, J_N), a positive real number b, and a function $\ell : (0, b) \to \mathbb{R}^{>0}$ such that (M, g_+, J_+) and (M, g_-, J_-) are obtained from the construction described in Example 4.19.*

The proof, explained in detail in Sections 5 and 6, goes roughly as follows: the main difficulty is to show that the complex structures J_+ and J_- necessarily commute. This is done in Theorem 4.17 using in an essential way the compactness assumption. When the complex dimension is at least 3, Theorem 4.17 also shows that $\mathrm{d}\varphi$ is preserved (up to sign) by J_+J_-. As a consequence, one can check that $J_+ + J_-$ defines a Hamiltonian 2-form of rank 1 with respect to both Kähler metrics g_+ and g_-. One can then either use the classification of compact manifolds with Hamiltonian forms obtained in [1] (which however is rather involved) or obtain the result in a simpler way by a geometric argument given in Proposition 4.21.

We now discuss the holonomy problem for compact proper lcK manifolds, that is, Problem P3, whose original motivation stems from [19].

By the Berger-Simons holonomy theorem, an lcK manifold (M^{2n}, g, J) either has reducible restricted holonomy representation, or is locally symmetric irreducible, or its restricted holonomy group $\mathrm{Hol}_0(M, g)$ is one of the following: $\mathrm{SO}(2n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{2})\mathrm{Sp}(1)$, $\mathrm{Spin}(7)$.

The restricted holonomy representation of a compact Riemannian manifold (M, g) is reducible if and only if the tangent bundle of a finite covering of M carries an oriented parallel (proper) distribution. We first show in Theorem 4.11 that a compact proper lcK manifold (M^{2n}, g, J) cannot carry a parallel distribution whose rank d satisfies $2 \leq d \leq 2n - 2$. The special case when this distribution is 1-dimensional was studied recently in [19], where the second named author described all compact proper lcK manifolds (M^{2n}, g, J) with $n \geq 3$ which carry a non-trivial parallel vector field. In Theorem 4.15 below we give an alternate proof of this classification, which is not only simpler, but also covers the missing case $n = 2$. This settles the reducible case.

The remaining possible cases given by the Berger-Simons theorem are either Einstein or Kähler (and gcK by Theorem 4.17), and thus fall into the previous classification results. Summarizing, we have the following classification result for the possible (restricted) holonomy groups of compact proper lcK manifolds:

Theorem 4.3 *Let (M^{2n}, g, J, θ), $n \geq 2$, be a compact proper lcK manifold with non-generic holonomy group $\mathrm{Hol}(M, g) \subsetneq \mathrm{SO}(2n)$. Then the following exclusive possibilities occur:*

1. *(M, g, J, θ) is strictly lcK, $\mathrm{Hol}(M, g) \simeq \mathrm{SO}(2n-1)$ and (M, g, J, θ) is Vaisman (that is, θ is parallel).*

2. *(M, g, J, θ) is gcK (that is, θ is exact) and either:*

   a) $n \geq 3$, $\mathrm{Hol}_0(M, g) \simeq \mathrm{U}(n)$, and a finite covering of (M, g, J, θ) is obtained by the Calabi Ansatz described in Example 4.19, or

   b) $n = 2$, $\mathrm{Hol}_0(M, g) \simeq \mathrm{U}(2)$ and M is ambikähler in the sense of [2], or

   c) $\mathrm{Hol}_0(M, g) \simeq \mathrm{SO}(2n-1)$ and a finite covering of (M, g, J, θ) is obtained by the construction described in Theorem 4.15.

2 Preliminaries on lcK manifolds

A locally conformally Kähler (lcK) manifold is a connected Hermitian manifold (M, g, J) of real dimension $2n \geq 4$ such that around each point, g is conformal to a metric which is Kähler with respect to J. The covariant derivative of J with respect to the Levi-Civita connection ∇ of g is determined by a closed 1-form θ (called the Lee form) via the formula (see e.g. [19]):

$$\nabla_X J = X \wedge J\theta + JX \wedge \theta, \qquad \forall\ X \in \mathrm{T}M. \tag{4.1}$$

Recall that if τ is any 1-form on M, $J\tau$ is the 1-form defined by $(J\tau)(X) := -\tau(JX)$ for every $X \in \mathrm{T}M$, and $X \wedge \tau$ denotes the endomorphism of $\mathrm{T}M$ defined by $(X \wedge \tau)(Y) := g(X, Y)\tau^\sharp - \tau(Y)X$. We will often identify 1-forms and vector fields via the metric g, which will also be denoted by $\langle \cdot, \cdot \rangle$ when there is no ambiguity.

Let $\Omega := g(J\cdot,\cdot)$ denote the associated 2-form of J. By (4.1), its exterior derivative and co-differential are given by

$$\mathrm{d}\Omega = 2\theta\wedge\Omega, \tag{4.2}$$

and

$$\delta\Omega = (2-2n)J\theta. \tag{4.3}$$

If $\theta\equiv 0$, the structure (g,J) is simply Kähler. If θ is not identically zero, then the lcK structure (g,J,θ) is called *proper.* If $\theta=\mathrm{d}\varphi$ is exact, then $\mathrm{d}(e^{-2\varphi}\Omega)=0$, so the conformally modified structure $(e^{-2\varphi}g,J)$ is Kähler, and the structure (g,J,θ) is called *globally conformally Kähler* (gcK). The lcK structure is called *strictly* lcK if the Lee form θ is not exact and *Vaisman* if θ is parallel with respect to the Levi-Civita connexion of g.

A typical example of strictly lcK manifold, which is actually Vaisman, is $\mathbb{S}^1\times\mathbb{S}^{2n-1}$, endowed with the complex structure induced by the diffeomorphism

$$(\mathbb{C}^n\setminus\{0\})/\mathbb{Z}\longrightarrow\mathbb{S}^1\times\mathbb{S}^{2n-1},\ \ [z]\longmapsto\left(e^{2\pi i\ln|z|},\frac{z}{|z|}\right),$$

where $[z]:=\{e^kz\in\mathbb{C}^n\setminus\{0\}\,|\,k\in\mathbb{Z}\}$. The Lee form of this lcK structure is the length element of $\mathbb{S}^1$, which is parallel.

Remark 4.4 *For each lcK manifold (M,g,J,θ) there exists a group homomorphism from $\pi_1(M)$ to $(\mathbb{R},+)$ which is trivial if and only if the structure is gcK. Indeed, $\pi_1(M)$ acts on the universal covering $\widetilde{M}$ of M, and preserves the induced lcK structure $(\tilde g,\tilde J,\tilde\theta)$. Since $\tilde\theta=\mathrm{d}\varphi$ is exact on $\widetilde{M}$, for every $\gamma\in\pi_1(M)$ we have $\mathrm{d}(\gamma^*\varphi)=\gamma^*(\mathrm{d}\varphi)=\gamma^*(\tilde\theta)=\tilde\theta=\mathrm{d}\varphi$, so there exists some real number c_γ such that $\gamma^*\varphi=\varphi+c_\gamma$. The map $\gamma\mapsto c_\gamma$ is clearly a group morphism from $\pi_1(M)$ to $(\mathbb{R},+)$, which is trivial if and only if θ is exact on M. This shows, in particular, that if $\pi_1(M)$ is finite, then every lcK structure on M is gcK.*

For later use, we express, for every lcK structure (g,J,θ), the action of the Riemannian curvature tensor of g on the Hermitian structure J.

Lemma 4.5 *The following formula holds for every vector fields X,Y on a lcK manifold (M,g,J,θ):*

$$\begin{aligned}&R_{X,Y}J=\theta(X)Y\wedge J\theta-\theta(Y)X\wedge J\theta-\theta(Y)JX\wedge\theta+\theta(X)JY\wedge\theta\\&-|\theta|^2Y\wedge JX+|\theta|^2X\wedge JY+Y\wedge J\nabla_X\theta+JY\wedge\nabla_X\theta-X\wedge J\nabla_Y\theta-JX\wedge\nabla_Y\theta.\end{aligned}\tag{4.4}$$

Proof. Taking X,Y parallel at the point where the computation is done and applying (4.1), we obtain:

$$\begin{aligned}R_{X,Y}J&=\nabla_X(Y\wedge J\theta+JY\wedge\theta)-\nabla_Y(X\wedge J\theta+JX\wedge\theta)\\&=Y\wedge(\nabla_XJ)(\theta)+(\nabla_XJ)(Y)\wedge\theta-X\wedge(\nabla_YJ)(\theta)-(\nabla_YJ)(X)\wedge\theta\\&\quad+Y\wedge J\nabla_X\theta+JY\wedge\nabla_X\theta-X\wedge J\nabla_Y\theta-JX\wedge\nabla_Y\theta,\end{aligned}$$

which gives (4.4) after a straightforward calculation using (4.1) again.

Let $\{e_i\}_{i=1,\dots,2n}$ be a local orthonormal basis of $\mathrm{T}M$. Substituting $Y = e_j$ in (4.4), taking the interior product with e_j and summing over $j = 1, \dots, 2n$ yields:

$$\sum_{j=1}^{2n}(R_{X,e_j}J)(e_j) = (2n-3)\big(\theta(X)J\theta - |\theta|^2 JX + J\nabla_X\theta\big) - \theta(JX)\theta - \nabla_{JX}\theta - JX\delta\theta, \tag{4.5}$$

since the sum $\sum_{j=1}^{2n} g(J\nabla_{e_j}\theta, e_j)$ vanishes, as $\nabla\theta$ is symmetric.

Corollary 4.6 *If the metric g of a compact lcK manifold (M, g, J, θ) is flat, then $\theta \equiv 0$.*

Proof. If the Riemannian curvature of g vanishes, (4.5) yields

$$0 = (2n-3)\big(\theta(X)J\theta - |\theta|^2 JX + J\nabla_X\theta\big) - \theta(JX)\theta - \nabla_{JX}\theta - JX\delta\theta.$$

We make the scalar product with JX in this equation for $X = e_j$, where $\{e_j\}_{j=1,\dots,2n}$ is a local orthonormal basis of $\mathrm{T}M$, and sum over $j = 1, \dots, 2n$ to obtain:

$$\begin{aligned} 0 &= (2n-3)\big(|\theta|^2 - 2n|\theta|^2 - \delta\theta\big) - |\theta|^2 + \delta\theta - 2n\delta\theta \\ &= -(2n-2)^2|\theta|^2 - 2(2n-2)\delta\theta. \end{aligned}$$

Since $n \geq 2$, this last equation yields $\delta\theta = (1-n)|\theta|^2$, which by Stokes' Theorem after integration over M gives $\theta \equiv 0$.

The following example shows that the corollary does not hold without the compactness assumption.

Example 4.7 *Consider the standard flat Kähler structure (g_0, J_0) on $M := \mathbb{C}^n \setminus \{0\}$. If r denotes the map $x \mapsto r(x) := |x|$, the conformal metric $g := r^{-4}g_0$ on M is gcK with respect to J_0, with Lee form $\theta = -2\mathrm{d}\ln r$. Moreover g is flat, being the pull-back of g_0 through the inversion $x \mapsto x/r^2$.*

3 Compact Einstein lcK manifolds

The purpose of this section is to classify compact Einstein proper lcK manifolds. We treat separately the two possible cases: non-negative and positive scalar curvature. In the non-negative case, we show that the Lee form must vanish, so the manifold is already Kähler. In the positive case, it follows that the manifold is globally conformally Kähler and one can use Maschler-Derdzinski's classification of conformally-Einstein Kähler metrics, for complex dimension $n \geq 3$, and the results of X. Chen, C. LeBrun and B. Weber [9] for complex surfaces.

Let (M, g, J, θ) be an lcK manifold. We denote by S the following symmetric 2-tensor:

$$S := \nabla\theta + \theta \otimes \theta, \tag{4.6}$$

identified with a symmetric endomorphism via the metric g.

Lemma 4.8 *On an lcK manifold (M, g, J, θ) with g Einstein, S commutes with J.*

Proof. Since the statement is local, we may assume without loss of generality that the Lee form is exact, $\theta = \mathrm{d}\varphi$, which means that $g_K := e^{-2\varphi}g$ is Kähler with respect to J. We denote the Einstein constant of g by λ.

The formula relating the Ricci tensors of conformally equivalent metrics [6, Theorem 1.159] reads:

$$\mathrm{Ric}^K = \mathrm{Ric}^g + 2(n-1)\left(\nabla^g \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) - \left(\Delta^g \varphi + 2(n-1) g(\mathrm{d}\varphi, \mathrm{d}\varphi)\right) g.$$

Since g^K is Kähler, $\mathrm{Ric}^K(J\cdot, J\cdot) = \mathrm{Ric}^K(\cdot, \cdot)$. Using this fact, together with $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ and $\mathrm{Ric}^g = \lambda g$ in the above formula, we infer:

$$\left(\nabla^g \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right)(J\cdot, J\cdot) = \left(\nabla^g \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right)(\cdot, \cdot), \tag{4.7}$$

which is equivalent to $SJ = JS$.

The main result of this section is the following:

Theorem 4.9 *If (M, g, J, θ) is a compact lcK manifold and g is Einstein with non-positive scalar curvature, then $\theta \equiv 0$, so (M, g, J) is a Kähler-Einstein manifold.*

Proof. Let $\{e_i\}_{i=1,\ldots,2n}$ be a local orthonormal basis which is parallel at the point where the computation is done. We denote by $\lambda \leq 0$ the Einstein constant of the metric g, so $\mathrm{Ric} = \lambda g$. The strategy of the proof is to apply the Bochner formula to the 1-forms θ and $J\theta$ in order to obtain a formula relating the Einstein constant, the co-differential of the Lee form and its square norm, which leads to a contradiction at a point where $\delta\theta + |\theta|^2$ attains its maximum if θ is not identically zero.

Let S denote as above the endomorphism $S = \nabla\theta + \theta \otimes \theta$. In particular, we have

$$S\theta = \nabla_\theta \theta + |\theta|^2 \theta = \frac{1}{2}\mathrm{d}|\theta|^2 + |\theta|^2\theta \tag{4.8}$$

and the trace of S is computed as follows

$$\mathrm{tr}(S) = |\theta|^2 - \delta\theta. \tag{4.9}$$

In the sequel, we use Lemma 4.8, ensuring that S commutes with J. We start by computing the covariant derivative of $J\theta$:

$$\begin{aligned}\nabla_X J\theta = (\nabla_X J)(\theta) + J(\nabla_X \theta) &\overset{(4.1)}{=} (X \wedge J\theta + JX \wedge \theta)(\theta) + J(SX - \theta(X)\theta) \\ &= JSX - J\theta(X)\theta - |\theta|^2 JX.\end{aligned} \tag{4.10}$$

The exterior differential of $J\theta$ is then given by the following formula:

$$\mathrm{d}J\theta = \sum_{i=1}^{2n} e_i \wedge \nabla_{e_i} J\theta = 2JS + \theta \wedge J\theta - 2|\theta|^2\Omega. \tag{4.11}$$

We further compute the Lie bracket between θ and $J\theta$ (viewed as vector fields):

$$[\theta, J\theta] = \nabla_\theta J\theta - \nabla_{J\theta}\theta \overset{(4.6),(4.10)}{=} JS\theta - |\theta|^2 J\theta - SJ\theta = -|\theta|^2 J\theta. \tag{4.12}$$

By (4.3), we have $\delta J\theta = 0$. Using the following identities:

$$\delta(\theta \wedge J\theta) = (\delta\theta)J\theta - \delta(J\theta)\theta - [\theta, J\theta] \overset{(4.12)}{=} (\delta\theta + |\theta|^2)J\theta, \tag{4.13}$$

$$\delta(|\theta|^2\Omega) = -J(\mathrm{d}|\theta|^2) + |\theta|^2\delta\Omega \overset{(4.3)}{=} -J(\mathrm{d}|\theta|^2) + (2-2n)|\theta|^2 J\theta, \tag{4.14}$$

we compute the Laplacian of $J\theta$:

$$\begin{aligned}\Delta J\theta &= \delta \mathrm{d} J\theta \overset{(4.11)}{=} \delta(2JS + \theta\wedge J\theta - 2|\theta|^2\Omega)\\ &\overset{(4.13),(4.14)}{=} 2\delta(JS) + (\delta\theta + |\theta|^2)J\theta + 2J(\mathrm{d}|\theta|^2) + 2(2n-2)|\theta|^2 J\theta\\ &= 2\delta(JS) + \delta\theta J\theta + 2J(\mathrm{d}|\theta|^2) + (4n-3)|\theta|^2 J\theta.\end{aligned} \tag{4.15}$$

We obtain the following formula for the connection Laplacian of $J\theta$:

$$\begin{aligned}\nabla^*\nabla J\theta &= -\sum_{i=1}^{2n}\nabla_{e_i}\nabla_{e_i}J\theta \overset{(4.10)}{=} -\sum_{i=1}^{2n}\nabla_{e_i}(JSe_i - J\theta(e_i)\theta - |\theta|^2 Je_i)\\ &= \delta(JS) + \nabla_{J\theta}\theta + J\mathrm{d}(|\theta|^2) + |\theta|^2\sum_{i=1}^{2n}(\nabla_{e_i}J)(e_i)\\ &\overset{(4.3)}{=} \delta(JS) + SJ\theta + J\mathrm{d}(|\theta|^2) + (2n-2)|\theta|^2 J\theta\\ &\overset{(4.8)}{=} \delta(JS) + |\theta|^2 J\theta + \frac{3}{2}J\mathrm{d}(|\theta|^2) + (2n-2)|\theta|^2 J\theta.\end{aligned} \tag{4.16}$$

The Bochner formula applied to $J\theta$ reads: $\Delta J\theta = \nabla^*\nabla J\theta + \mathrm{Ric}(J\theta)$. Substituting in this formula equations (4.15) and (4.16) yields

$$-\delta(JS) = (\delta\theta)J\theta + \frac{1}{2}J(\mathrm{d}|\theta|^2) + 2(n-1)|\theta|^2 J\theta - \lambda J\theta,$$

which, after applying $-J$ on both sides, reads:

$$J\delta(JS) = (\delta\theta)\theta + \frac{1}{2}\mathrm{d}|\theta|^2 + 2(n-1)|\theta|^2\theta - \lambda\theta. \tag{4.17}$$

The connection Laplacian of θ is computed as follows:

$$\nabla^*\nabla\theta = -\sum_{i=1}^{2n}\nabla_{e_i}\nabla_{e_i}\theta = -\sum_{i=1}^{2n}\nabla_{e_i}(Se_i - \theta(e_i)\theta) = \delta S - (\delta\theta)\theta + \frac{1}{2}\mathrm{d}|\theta|^2. \tag{4.18}$$

The Bochner formula applied to θ yields $\Delta\theta = \nabla^*\nabla\theta + \mathrm{Ric}(\theta)$, which by (4.18) is equivalent to

$$\delta S = (\delta\theta)\theta - \frac{1}{2}\mathrm{d}|\theta|^2 - \lambda\theta + \mathrm{d}\delta\theta. \tag{4.19}$$

On the other hand, we have:

$$
\begin{aligned}
\delta(JS) &= -\sum_{i=1}^{2n}(\nabla_{e_i}JS)(e_i) = -\sum_{i=1}^{2n}(\nabla_{e_i}J)(Se_i) - \sum_{i=1}^{2n}J(\nabla_{e_i}S)(e_i)\\
&\overset{(4.1)}{=} -\sum_{i=1}^{2n}(e_i \wedge J\theta + Je_i \wedge \theta)(Se_i) + J(\delta S)\\
&= -\mathrm{tr}(S)J\theta + 2JS\theta + J(\delta S) \overset{(4.8),(4.9)}{=} (\delta\theta)J\theta + J(\mathrm{d}|\theta|^2) + |\theta|^2 J\theta + J(\delta S).
\end{aligned}
$$

Applying J to this equality yields

$$
-J\delta(JS) - \delta S = (\delta\theta)\theta + (\mathrm{d}|\theta|^2) + |\theta|^2\theta. \tag{4.20}
$$

Summing up (4.17), (4.19) and (4.20), we obtain:

$$
3(\delta\theta)\theta - 2\lambda\theta + \mathrm{d}\delta\theta + \mathrm{d}|\theta|^2 + (2n-1)|\theta|^2\theta = 0. \tag{4.21}
$$

Denoting by f the function $f := \delta\theta + |\theta|^2$, the equality (4.21) is equivalent to:

$$
\mathrm{d}f = (2\lambda - 3f + (4-2n)|\theta|^2)\theta. \tag{4.22}
$$

We argue by contradiction and assume that θ is not identically zero. Therefore, the integral of f over M is positive. As M is compact, there exists $p_0 \in M$ at which f attains its maximum, $f(p_0) > 0$. In particular, we have $(\mathrm{d}f)_{p_0} = 0$ and $(\Delta f)(p_0) \geq 0$. Applying (4.22) at the point p_0 yields that $\theta_{p_0} = 0$, because $2\lambda - 3f(p_0) + (4-2n)|\theta_{p_0}|^2 < 0$. From the definition of f, it follows that $\delta\theta(p_0) > 0$.

On the other hand, taking the co-differential of (4.22), we obtain:

$$
\Delta f = (2\lambda - 3f + (4-2n)|\theta|^2)\delta\theta + 3\theta(f) + (2n-4)\theta(|\theta|^2).
$$

Evaluating at p_0 leads to a contradiction, since the left-hand side is non-negative and the right-hand side is negative, as $\theta_{p_0} = 0$ and $(2\lambda - 3f(p_0))\delta\theta(p_0) < 0$. Thus, $\theta \equiv 0$.

Note that in complex dimension $n = 2$, C. LeBrun [16] showed, by extending results of A. Derdzinski [10], that a Hermitian non-Kähler Einstein metric on a compact complex surface is necessarily conformal to a Kähler metric and has positive scalar curvature. In particular, this result implies the statement of Theorem 4.9 for complex surfaces. However, the method of our proof works in all dimensions.

If (M^{2n}, g, J, θ) is a compact lcK manifold and g is Einstein with positive scalar curvature, then by Myers' Theorem and Remark 4.4, (M, g, J) is gcK. The classification of conformally Kähler compact Einstein manifolds in complex dimension $n \geq 3$ has been obtained by A. Derdzinski and G. Maschler in a series of three papers [11, 12, 13]. They showed that the only examples are given by the construction of L. Bérard-Bergery, [5]. In complex dimension $n = 2$, the only compact complex surfaces which might admit proper gcK Einstein metrics are the blow-up of $\mathbb{C}P^2$ at one, two or three points in general position, according to a result of C. LeBrun, [16, Theorem A]. Moreover, in the one point case, he showed that, up to rescaling and isometry, the only such metric is the well-known Page metric, [21]. The existence of a Hermitian Einstein metric on the blow-up of $\mathbb{C}P^2$ at two different points was proven by X. Chen, C. LeBrun and B. Weber in [9].

Theorem 4.9 and the above remarks complete the proof of Theorem 4.1.

4 The holonomy problem for compact lcK manifolds

The aim of this section is to study compact lcK manifolds (M, g, J, θ) of complex dimension $n \geq 2$ with non-generic holonomy group: $\mathrm{Hol}_0(M, g) \subsetneq \mathrm{SO}(2n)$. By the Berger-Simons holonomy theorem, the following exclusive possibilities may occur:

- The restricted holonomy group $\mathrm{Hol}_0(M, g)$ is reducible;
- $\mathrm{Hol}_0(M, g)$ is irreducible and (M, g) is locally symmetric;
- M is not locally symmetric, and $\mathrm{Hol}_0(M, g)$ belongs to the following list: $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n/2)$, $\mathrm{Sp}(n/2)\mathrm{Sp}(1)$, $\mathrm{Spin}(7)$ (for $n = 4$).

4.1 The reducible case

In this section we classify the compact lcK manifolds with reducible restricted holonomy. We start with the following result (for a proof see for instance the first part of the proof of [4, Theorem 4.1]):

Lemma 4.10 *If (M, g) is a compact Riemannian manifold with $\mathrm{Hol}_0(M, g)$ reducible, then there exists a finite covering $\overline{M}$ of M, such that $\mathrm{Hol}(\overline{M}, \bar{g})$ is reducible.*

Let now (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ with $\mathrm{Hol}_0(M, g)$ reducible. Lemma 4.10 shows that by replacing M with some (compact) finite covering $\overline{M}$, and by pulling back the lcK structure to $\overline{M}$, one may assume that the tangent bundle can be decomposed as $\mathrm{T}M = D_1 \oplus D_2$, where D_1 and D_2 are two parallel orthogonal oriented distributions of rank n_1, respectively n_2, with $2n = n_1 + n_2$. We first show that the case $n_1 \geq 2$ and $n_2 \geq 2$ is impossible if the lcK structure is proper.

Theorem 4.11 *Let (M, g, J, θ) be a compact lcK manifold of complex dimension $n \geq 2$. If there exist two orthogonal parallel oriented distributions D_1 and D_2 of rank $n_1 \geq 2$ and $n_2 \geq 2$ such that $\mathrm{T}M = D_1 \oplus D_2$, then $\theta \equiv 0$.*

Proof. Since the arguments for $n = 2$ and $n \geq 3$ are of different nature, we treat the two cases separately. Consider first the case of complex dimension $n = 2$. Then both distributions D_1 and D_2 have rank 2, and their volume forms Ω_1 and Ω_2 define two Kähler structures on M compatible with g by the formula $g(I_\pm \cdot, \cdot) = \Omega_1 \pm \Omega_2$. Using the case $n = 2$ in Theorem 4.17 below, we deduce that J commutes with I_+ and with I_-. In particular, J preserves the ± 1 eigenspaces of $I_+ I_-$, which are exactly the distributions

D_1 and D_2. Since J is also orthogonal, its restriction to D_1 and D_2 coincides up to sign with the restriction of I_+ to D_1 and D_2. Thus $J = \pm I_+$ or $J = \pm I_-$. In each case, the structure (g, J) is Kähler, so $\theta \equiv 0$.

We consider now the case $n \geq 3$. Let $\theta = \theta_1 + \theta_2$ be the corresponding splitting of the Lee form. We fix a local orthonormal basis $\{e_i\}_{i=1,\dots,2n}$, which is parallel at the point where the computation is done and denote by e_i^a the projection of e_i onto D_a, for $a \in \{1, 2\}$.

The exterior differential and Ω split with respect to the decomposition of the tangent bundle as follows: $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$ and $\Omega = \Omega_{11} + 2\Omega_{12} + \Omega_{22}$, where for $a, b \in \{1, 2\}$ we define:

$$\mathrm{d}_a := \sum_{i=1}^{2n} e_i^a \wedge \nabla_{e_i^a}, \quad \Omega_{ab} := \frac{1}{2}\sum_{i=1}^{2n} e_i^a \wedge (Je_i)^b = \frac{1}{2}\sum_{i=1}^{2n} e_i^a \wedge (Je_i^a)^b.$$

Lemma 4.12 *With the above notation, for any vector fields $X_1 \in D_1$ and $X_2 \in D_2$, the following relations hold:*

$$\nabla_{X_1}\theta_2 = -\theta_1(X_1)\theta_2, \quad \nabla_{X_2}\theta_1 = -\theta_2(X_2)\theta_1. \tag{4.23}$$

Proof. Note that $\mathrm{d}\theta = 0$ implies $\mathrm{d}_a\theta_b + \mathrm{d}_b\theta_a = 0$, for all $a, b \in \{1, 2\}$. For $c \in \{1, 2\}$ we compute:

$$\begin{aligned}
\mathrm{d}_c\Omega_{ab} &\overset{(4.1)}{=} \frac{1}{2}\sum_{i,j=1}^{2n} e_j^c \wedge e_i^a \wedge \big(\langle e_j^c, e_i\rangle J\theta - \langle J\theta, e_i\rangle e_j^c + \langle Je_j^c, e_i\rangle\theta - \langle \theta, e_i\rangle Je_j^c\big)^b \\
&= \frac{1}{2}\sum_{i=1}^{2n}\Big(e_i^c \wedge e_i^a \wedge J\theta^b - (Je_i)^c \wedge e_i^a \wedge \theta_b - e_i^c \wedge \theta_a \wedge (Je_i^c)^b\Big) \\
&= \Omega_{ac} \wedge \theta_b + \Omega_{cb} \wedge \theta_a,
\end{aligned}$$

so for all $a, b, c \in \{1, 2\}$ we have

$$\mathrm{d}_c\Omega_{ab} = \Omega_{ac} \wedge \theta_b + \Omega_{cb} \wedge \theta_a. \tag{4.24}$$

Using the fact that $\mathrm{d}_c^2 = 0$ for $c \in \{1, 2\}$, we obtain

$$0 = \mathrm{d}_c^2\Omega_{ab} = \Omega_{ca} \wedge (\mathrm{d}_c\theta_b + \theta_c \wedge \theta_b) + \Omega_{cb} \wedge (\mathrm{d}_c\theta_a + \theta_c \wedge \theta_a). \tag{4.25}$$

For $c = a \neq b$ in (4.25), we get $\Omega_{cc}\wedge(\mathrm{d}_c\theta_b+\theta_c\wedge\theta_b) = 0$ and for $a = b$: $\Omega_{cb}\wedge(\mathrm{d}_c\theta_b+\theta_c\wedge\theta_b) = 0$. Summing up, we obtain that $\Omega \wedge (\mathrm{d}_c\theta_b + \theta_c \wedge \theta_b) = 0$, which by the injectivity of $\Omega \wedge \cdot$ on manifolds of real dimension greater than 4, implies that $\mathrm{d}_c\theta_b = -\theta_c \wedge \theta_b$. Applying this identity for $b \neq c$ to $X_c \in D_c$ and $X_b \in D_b$ yields (4.23).

The symmetries of the Riemannian curvature tensor imply that $R_{X_1,X_2} = 0$, and thus $R_{X_1,X_2}J = [R_{X_1,X_2}, J] = 0$, for every $X_1 \in D_1$ and $X_2 \in D_2$.

Using (4.4) for $X := X_1$ and $Y := X_2$ and applying Lemma 4.12, we obtain:

$$\begin{aligned}
0 = {}&\langle X_1, \theta_1\rangle X_2 \wedge J\theta_1 - \langle X_2, \theta_2\rangle X_1 \wedge J\theta_2 - \langle X_2, \theta_2\rangle JX_1 \wedge \theta_2 + \langle X_1, \theta_1\rangle JX_2 \wedge \theta_1 \\
&-|\theta|^2X_2\wedge JX_1+|\theta|^2X_1\wedge JX_2+X_2\wedge J\nabla_{X_1}\theta_1+JX_2\wedge\nabla_{X_1}\theta_1-X_1\wedge J\nabla_{X_2}\theta_2-JX_1\wedge\nabla_{X_2}\theta_2,
\end{aligned} \tag{4.26}$$

for every $X_1 \in D_1$ and $X_2 \in D_2$.

Lemma 4.13 *The following formula holds:*

$$\nabla_{X_1}\theta_1 = -\langle X_1, \theta_1 \rangle \theta_1 + \frac{1}{n_1}\left(|\theta_1|^2 - \delta\theta_1\right) X_1, \quad \forall X_1 \in D_1. \tag{4.27}$$

Proof.
Let $\mathcal{U}$ denote the open set $\mathcal{U} := \{x \in M \mid (JD_2)_x \not\subset (D_1)_x\}$. By continuity, it is enough to prove the result on the open sets $M \setminus \overline{\mathcal{U}}$ and $\mathcal{U}$.

Let $\mathcal{O}$ be some open subset of $M \setminus \overline{\mathcal{U}}$, *i.e.* at every point x of $\mathcal{O}$ the inclusion $(JD_2)_x \subset (D_1)_x$ holds. On $\mathcal{O}$, let X be some vector field and Y_2, Z_2 vector fields tangent to D_2. By assumption, we have $JY_2 \in D_1$, hence $\nabla_X JY_2 \in D_1$ and $\nabla_X Y_2 \in D_2$, thus $J\nabla_X Y_2 \in D_1$. Applying (4.1), we obtain

$$\begin{aligned} 0 &= \langle \nabla_X JY_2, Z_2 \rangle - \langle J\nabla_X Y_2, Z_2 \rangle = \langle (\nabla_X J)Y_2, Z_2 \rangle \\ &= \langle X, Y_2 \rangle (J\theta)(Z_2) - \langle X, Z_2 \rangle (J\theta)(Y_2) - \langle X, JY_2 \rangle \theta(Z_2) + \langle X, JZ_2 \rangle \theta(Y_2). \end{aligned}$$

Since $n_2 \geq 2$, for any $Y_2 \in D_2$ there exists a non-zero $Z_2 \in D_2$ orthogonal to Y_2. Taking $X = JZ_2 \in D_1$ in the above formula yields $\theta(Y_2) = 0$. This shows that $\theta_2 = 0$, so $\theta = \theta_1$. Taking $X = Z_2 \in D_2$ in the above formula yields $\theta_1(JY_2) = 0$, for all $Y_2 \in D_2$. Substituting into (4.26), we obtain for all $X_1 \in D_1$ and $Y_2 \in D_2$:

$$\begin{aligned} \langle X_1, \theta_1 \rangle Y_2 \wedge J\theta_1 + \langle X_1, \theta_1 \rangle JY_2 \wedge \theta_1 - |\theta_1|^2 Y_2 \wedge JX_1 + |\theta_1|^2 X_1 \wedge JY_2 \\ + Y_2 \wedge J\nabla_{X_1}\theta_1 + JY_2 \wedge \nabla_{X_1}\theta_1 = 0, \end{aligned} \tag{4.28}$$

Let us now consider the decomposition $D_1 = JD_2 \oplus D_1'$, where D_1' denotes the orthogonal complement of JD_2 in D_1. Note that D_1' is J-invariant, since it is also the orthogonal complement in $\mathrm{T}M$ of the J-invariant distribution $D_2 \oplus JD_2$. Let $X_1 = JV_2 + V_1$ and $\nabla_{X_1}\theta_1 = JW_2 + W_1$ be the decomposition of X_1, respectively of $\nabla_{X_1}\theta_1$, with respect to this splitting, *i.e.* $V_2, W_2 \in D_2$ and $V_1, W_1 \in D_1'$. As shown above, θ_1 vanishes on JD_2, meaning that $\theta_1 \in D_1'$.

Taking the trace with respect to Y_2 in (4.28) yields

$$n_2 \langle X_1, \theta_1 \rangle J\theta_1 + |\theta_1|^2[(n_2 - 1)V_2 - n_2 JV_1] + n_2 JW_1 - (n_2 - 1)W_2 = 0, \tag{4.29}$$

which further implies, by projecting onto D_2 and D_1', that $W_1 = -\langle X_1, \theta_1 \rangle \theta_1 + |\theta_1|^2 V_1$ and $W_2 = |\theta_1|^2 V_2$. Hence, $\nabla_{X_1}\theta_1 = |\theta_1|^2 X_1 - \langle X_1, \theta_1 \rangle \theta_1$, which in particular implies $\delta\theta_1 = (1 - n_1)|\theta_1|^2$, proving (4.27) on $M \setminus \overline{\mathcal{U}}$.

We further show that the formula (4.27) holds on $\mathcal{U}$. At every point x of $\mathcal{U}$ there exist vectors $X_2, Y_2 \in (D_2)_x$ such that $X_2 \perp Y_2$ and $\langle Y_2, JX_2 \rangle \neq 0$. Indeed, by definition there exists $Y_2 \in (D_2)_x$ such that $JY_2 \notin D_1$, and we can take X_2 to be the D_2-projection of JY_2.

For any vector $X_1 \in (D_1)_x$ we take the scalar product with $X_1 \wedge Y_2$ in (4.26) and obtain:

$$\begin{aligned} \langle JX_2, Y_2 \rangle \left(\langle \nabla_{X_1}\theta_1, X_1 \rangle + |\langle X_1, \theta_1 \rangle|^2 \right) = \\ - |X_1|^2 \left(\langle X_2, \theta_2 \rangle \langle J\theta_2, Y_2 \rangle - |\theta|^2 \langle JX_2, Y_2 \rangle + \langle J\nabla_{X_2}\theta_2, Y_2 \rangle \right). \end{aligned} \tag{4.30}$$

We thus get $\langle \nabla_{X_1}\theta_1, X_1 \rangle + |\langle X_1, \theta_1(x) \rangle|^2 = f_1(x)|X_1|^2$, for every $X_1 \in (D_1)_x$, where the real number $f_1(x)$ does not depend on X_1. By polarization, we obtain:

$$\nabla_{X_1}\theta_1 = -\langle X_1, \theta_1(x) \rangle \theta_1(x) + f_1(x) X_1, \quad \forall X_1 \in (D_1)_x. \tag{4.31}$$

Taking the trace with respect to X_1 in this formula and using (4.23) we obtain $(\delta\theta_1)_x = |\theta_1(x)|^2 - n_1 f_1(x)$, whence:

$$f_1(x) = \frac{1}{n_1}\left(|\theta_1|^2 - \delta\theta_1\right)(x), \qquad \forall x \in \mathcal{U}. \tag{4.32}$$

From (4.31) and (4.32) we obtain (4.27) on $\mathcal{U}$. This proves the lemma.

A similar argument yields

$$\nabla_{X_2}\theta_2 = -\langle X_2, \theta_2 \rangle \theta_2 + \frac{1}{n_2}\left(|\theta_2|^2 - \delta\theta_2\right) X_2, \quad \forall X_2 \in D_2. \tag{4.33}$$

Substituting (4.27) and (4.33) into (4.26), we obtain

$$\left(\frac{1}{n_1}\left(|\theta_1|^2 - \delta\theta_1\right) + \frac{1}{n_2}\left(|\theta_2|^2 - \delta\theta_2\right) - |\theta|^2\right)(X_2 \wedge JX_1 - X_1 \wedge JX_2) = 0, \ \forall X_j \in D_j.$$

Note that for every $X_1 \in D_1$, $X_2 \in D_2$ the two-forms $X_2 \wedge JX_1$ and $X_1 \wedge JX_2$ are mutually orthogonal. So, choosing X_1 non-collinear to JX_2 (which is possible as $n_1 \geq 2$), the 2-form appearing in the previous formula is non-zero. Hence, we necessarily have

$$\frac{1}{n_1}\left(|\theta_1|^2 - \delta\theta_1\right) + \frac{1}{n_2}\left(|\theta_2|^2 - \delta\theta_2\right) - |\theta|^2 = 0.$$

Integrating this relation over M, we get

$$\int_M |\theta|^2 \mathrm{d}\mu_g = \frac{1}{n_1}\int_M |\theta_1|^2 \mathrm{d}\mu_g + \frac{1}{n_2}\int_M |\theta_2|^2 \mathrm{d}\mu_g.$$

Since $|\theta|^2 = |\theta_1|^2 + |\theta_2|^2$, we obtain $\left(1 - \frac{1}{n_1}\right)\int_M |\theta_1|^2 \mathrm{d}\mu_g + \left(1 - \frac{1}{n_2}\right)\int_M |\theta_2|^2 \mathrm{d}\mu_g = 0$. As $n_1, n_2 \geq 2$, it follows that $\theta \equiv 0$. This concludes the proof of the theorem.

Remark 4.14 *For every $n \geq 2$, the tangent bundle $\mathrm{T}(\mathbb{C}^n \setminus \{0\})$ endowed with the flat metric g defined in Example 4.7 can be written as an orthogonal direct sum of two parallel distributions of ranks at least 2 in infinitely many ways, but the gcK structure (g, J_0) on $\mathbb{C}^n \setminus \{0\}$ has non-vanishing Lee form $\theta = -2\ln r$. The compactness assumption in Theorem 4.11 is thus necessary.*

It remains to consider the case when one of the two parallel distributions has rank 1. By taking a further double covering if necessary, we may assume that this 1-dimensional distribution is oriented, and thus spanned by a globally defined unit vector field. Since the distribution is preserved by ∇, this vector field is clearly parallel. This case was studied by the second named author in [19, Theorem 3.5] for $n \geq 3$. We will give here a simpler proof of his result, which also extends it to the missing case $n = 2$.

Theorem 4.15 (*cf.* [19, Theorem 3.5]) *Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ admitting a non-trivial parallel vector field V. Then, the following exclusive possibilities occur:*

(i) The Lee form θ is a non-zero constant multiple of $V^\flat$, so M is a Vaisman manifold.

(ii) The Lee form θ is exact, so (M, g, Ω, θ) is gcK, and there exists a complete simply connected Kähler manifold (N, g_N, Ω_N) of real dimension $2n-2$, a smooth non-constant real function $c : \mathbb{R} \to \mathbb{R}$ and a discrete co-compact group Γ acting freely and totally discontinuously on $\mathbb{R}^2 \times N$, preserving the metric $\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)} g_N$, the Hermitian 2-form $\mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N$ and the vector fields ∂_s and ∂_t, such that M is diffeomorphic to $\Gamma\backslash(\mathbb{R}^2 \times N)$, and the structure (g, Ω, θ) corresponds to $(\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)} g_N, \mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N, \mathrm{d}c)$ through this diffeomorphism.

Proof. Let V be a parallel vector field of unit length on M. We identify as usual 1-forms with vectors using the metric g and decompose the Lee form as $\theta = aV + bJV + \theta_0$, where $a := \langle \theta, V \rangle$, $b := \langle \theta, JV \rangle$ and θ_0 is orthogonal onto V and JV. We compute:

$$\delta\theta = -V(a) - JV(b) + b\delta JV + \delta\theta_0.$$

On the other hand, we have:

$$\begin{aligned}
\delta JV &= -\sum_{i=1}^{2n} \langle (\nabla_{e_i} J)V, e_i \rangle \overset{(4.1)}{=} \sum_{i=1}^{2n} -\langle (e_i \wedge J\theta + Je_i \wedge \theta)(V), e_i \rangle \\
&= (2-2n)\langle \theta, JV \rangle = (2-2n)b,
\end{aligned}$$

which yields

$$\delta\theta = -V(a) - JV(b) + (2-2n)b^2 + \delta\theta_0. \tag{4.34}$$

Replacing X by V in (4.5) and using that V is parallel, we obtain:

$$(2n-3)\left(aJ\theta - |\theta|^2 JV + J\nabla_V\theta\right) - b\theta - \nabla_{JV}\theta - JV\delta\theta = 0.$$

Taking the scalar product with JV yields

$$(2n-3)\left(a^2 - |\theta|^2 + \langle \nabla_V\theta, V \rangle\right) - b^2 - \langle \nabla_{JV}\theta, JV \rangle - \delta\theta = 0. \tag{4.35}$$

Further, we compute

$$\langle \nabla_V\theta, V \rangle = V(\langle \theta, V \rangle) = V(a),$$

$$\langle \nabla_{JV}\theta, JV \rangle = JV(b) - \langle \theta, (\nabla_{JV}J)V \rangle \overset{(4.1)}{=} JV(b) - \langle \theta, bJV + aV - \theta \rangle = JV(b) + |\theta_0|^2,$$

which together with (4.34) and (4.35) imply that

$$(2n-2)(V(a) - |\theta_0|^2) = \delta\theta_0.$$

Integrating over M, we obtain $\int_M |\theta_0|^2 = 0$, because $\int_M V(a)\mathrm{d}\mu_g = \int_M a\delta V \mathrm{d}\mu_g = 0$, as V is parallel. Hence, $\theta_0 = 0$, showing that $\theta = aV + bJV$.

Claim. The function a is constant and $ab = 0$.

Proof of the Claim. Equation (4.1) yields

$$\nabla_X JV = \langle X, V \rangle(-bV + aJV) + bX - \langle X, JV \rangle(aV + bJV) - aJX, \tag{4.36}$$

which allows us to compute the exterior differential of JV, as follows:

$$\mathrm{d}JV = 2a(V \wedge JV - \Omega). \tag{4.37}$$

From the fact that θ is closed and V is parallel, we obtain

$$0 = \mathrm{d}a \wedge V + \mathrm{d}b \wedge JV + b\,\mathrm{d}JV = \mathrm{d}a \wedge V + \mathrm{d}b \wedge JV + 2ab(V \wedge JV - \Omega),$$

which implies that $ab = 0$, for instance, by taking the scalar product with $X \wedge JX$ for some vector field X orthogonal to V and JV. In particular, we have

$$\mathrm{d}a \wedge V + \mathrm{d}b \wedge JV = 0. \tag{4.38}$$

Differentiating again (4.37) yields

$$0 = \mathrm{d}a\wedge(V\wedge JV-\Omega)+a(-V\wedge\mathrm{d}JV-\mathrm{d}\Omega) = \mathrm{d}a\wedge(V\wedge JV-\Omega)-2abJV\wedge\Omega = \mathrm{d}a\wedge(V\wedge JV-\Omega),$$

which together with (4.38) shows that the function a is constant, thus proving the claim.

If a is non-zero, the second part of the claim shows that $b \equiv 0$, so $\theta = aV$ is parallel and (M, g, J, θ) is Vaisman.

If $a = 0$, Equation (4.36) becomes:

$$\nabla_X JV = b(X - \langle X, V \rangle V - \langle X, JV \rangle JV).$$

We conclude that in this case the metric structure on M is given as in (ii) by applying Lemma 3.3 and Lemma 3.4 in [19].

4.2 The irreducible locally symmetric case

In this section we show the following result:

Proposition 4.16 *Every compact irreducible locally symmetric lcK manifold (M^{2n}, g, J, θ) has vanishing Lee form.*

Proof. An irreducible locally symmetric space is Einstein. If the scalar curvature of M is non-positive, the result follows directly from Theorem 4.9.

Assume now that M has positive scalar curvature. By Myers' Theorem and Remark 4.4, (M, g, J) is gcK, so $\theta = \mathrm{d}\varphi$ for some function φ, and $g_K := e^{-2\varphi}g$ is a Kähler metric. Let X be a Killing vector field of g. Then X is a conformal Killing vector field of the metric g_K. By a result of Lichnerowicz [17] and Tashiro [22], every conformal Killing vector field with respect to a Kähler metric on a compact manifold is Killing. This shows that X is a Killing vector field for both conformal metrics g and g_K, hence X preserves the conformal factor, *i.e.* $X(\varphi) = 0$. As (M, g) is homogeneous and $X(\varphi) = 0$ for each Killing vector field X of g, it follows that the function φ is constant. Thus $\theta = \mathrm{d}\varphi = 0$.

4.3 Compact irreducible lcK manifolds with special holonomy

We finally consider compact lcK manifolds (M, g, J, θ) of complex dimension $n \geq 2$, whose restricted holonomy group $\mathrm{Hol}_0(M, g)$ is in the Berger list. The following cases occur:

If $\mathrm{Hol}_0(M, g)$ is one of $\mathrm{SU}(n)$, $\mathrm{Sp}(n/2)$, or $\mathrm{Spin}(7)$ (for $n = 4$), the metric g is Ricci-flat and $\theta \equiv 0$ by Theorem 4.9.

If $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/2)\mathrm{Sp}(1)$, the metric g is quaternion-Kähler, hence Einstein with either positive or negative scalar curvature. In the negative case one has $\theta \equiv 0$ by Theorem 4.9. On the other hand, P. Gauduchon, A. Moroianu and U. Semmelmann, have shown in [14], that the only compact quaternion-Kähler manifolds of positive scalar curvature which carry an almost complex structure are the complex Grassmanians of 2-planes, which are symmetric, thus again $\theta \equiv 0$ by Proposition 4.16.

The case $\mathrm{Hol}_0(M, g) = \mathrm{U}(n)$ is more involved and will be treated in the next sections.

5 Kähler structures on lcK manifolds

In this section we study the last case left open in the previous section, namely compact proper lcK manifolds (M, g, J, θ) whose restricted holonomy group is equal to $\mathrm{U}(n)$. We will see that there are examples of such structures, but they cannot be strictly lcK. In particular, the Riemannian metric g of a compact strictly lcK manifold (M, g, J, θ) cannot be Kähler with respect to any complex structure on M.

The universal covering $(\widetilde{M}, \tilde{g})$ has holonomy $\mathrm{Hol}(\widetilde{M}, \tilde{g}) = \mathrm{U}(n)$, so $\tilde{g}$ is Kähler with respect to some complex structure $\tilde{I}$. Every deck transformation γ of $\widetilde{M}$ is an isometry of $\tilde{g}$, so $\gamma^* \tilde{I}$ is parallel with respect to the Levi-Civita connection of $\tilde{g}$. As $\mathrm{Hol}(\widetilde{M}, \tilde{g}) = \mathrm{U}(n)$, we necessarily have $\gamma^* \tilde{I} = \pm \tilde{I}$ for every $\gamma \in \pi_1(M) \subset \mathrm{Iso}(\widetilde{M})$. The group of $\tilde{I}$-holomorphic deck transformations is thus a subgroup of index at most 2 of $\pi_1(M)$, showing that after replacing M with some double covering if necessary, there exists an integrable complex structure I, such that (M, g, I) is a Kähler manifold.

Theorem 4.17 *Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ carrying a complex structure I, such that (M, g, I) is a Kähler manifold. Then I commutes with J and (M, g, J, θ) is globally conformally Kähler.*

Proof. The Riemannian curvature tensor of (M, g) satisfies $R_{X,Y} = R_{IX,IY}$, so in particular we have $R_{X,Y}J = R_{IX,IY}J$, for all vector fields X and Y. Using (4.4), this identity implies that

$$\begin{aligned}
\langle X, \theta \rangle Y \wedge J\theta - \langle Y, \theta \rangle X \wedge J\theta - \langle Y, \theta \rangle JX \wedge \theta + \langle X, \theta \rangle JY \wedge \theta - |\theta|^2 Y \wedge JX + |\theta|^2 X \wedge JY \\
+ Y \wedge J\nabla_X \theta + JY \wedge \nabla_X \theta - X \wedge J\nabla_Y \theta - JX \wedge \nabla_Y \theta \\
= \langle IX, \theta \rangle IY \wedge J\theta - \langle IY, \theta \rangle IX \wedge J\theta - \langle IY, \theta \rangle JIX \wedge \theta + \langle IX, \theta \rangle JIY \wedge \theta \\
- |\theta|^2 IY \wedge JIX + |\theta|^2 IX \wedge JIY + IY \wedge J\nabla_{IX}\theta + JIY \wedge \nabla_{IX}\theta - IX \wedge J\nabla_{IY}\theta - JIX \wedge \nabla_{IY}\theta,
\end{aligned}$$

for all vector fields X, Y. Let $\{e_i\}_{i=1,\dots,2n}$ be a local orthonormal basis of $\mathrm{T}M$, which is parallel at the point where the computation is done. Taking the interior product with X in the above identity and summing over $X = e_i$, we obtain

$$\begin{aligned}(4-2n)\langle Y, \theta\rangle J\theta + \langle JY, \theta\rangle\theta + (2n-4)|\theta|^2 JY + (4-2n)J\nabla_Y\theta + \nabla_{JY}\theta + \delta\theta JY \\ = \langle IY, \theta\rangle IJ\theta - \langle IJ\theta, \theta\rangle IY - \langle IY, \theta\rangle \mathrm{tr}(JI)\theta + \langle IY, \theta\rangle JI\theta + \langle \theta, IJIY\rangle\theta \\ +|\theta|^2\mathrm{tr}(IJ)IY - |\theta|^2 IJIY - \langle e_i, J\nabla_{Ie_i}\theta\rangle IY + IJ\nabla_{IY}\theta + \nabla_{IJIY}\theta - \mathrm{tr}(JI)\nabla_{IY}\theta + JI\nabla_{IY}\theta.\end{aligned} \tag{4.39}$$

Substituting $Y = e_j$ in (4.39), taking the scalar product with Je_j and summing over $j = 1, \dots, 2n$ yields:

$$\begin{aligned}(4n^2 - 10n + 6)|\theta|^2 + (4n-6)\delta\theta = -2\mathrm{tr}(IJ)\langle I\theta, J\theta\rangle - 2\langle IJ\theta, JI\theta\rangle - (\mathrm{tr}(IJ))^2|\theta|^2 \\ + \mathrm{tr}(IJIJ)|\theta|^2 + 2\mathrm{tr}(IJ)\sum_{i=1}^{2n}\langle IJ\nabla_{e_i}\theta, e_i\rangle - 2\sum_{i=1}^{2n}\langle IJIJ\nabla_{e_i}\theta, e_i\rangle.\end{aligned} \tag{4.40}$$

By a straightforward computation, using (4.1) and the fact that $\nabla\theta$ is a symmetric endomorphism, we have the following identities:

$$\mathrm{tr}(IJ)\sum_{i=1}^{2n}\langle e_i, IJ\nabla_{e_i}\theta\rangle = -\delta(\mathrm{tr}(IJ)JI\theta) + (2n-2)\mathrm{tr}(IJ)\langle I\theta, J\theta\rangle - 2\langle IJ\theta, JI\theta\rangle + 2|\theta|^2,$$

$$\sum_{i=1}^{2n}\langle e_i, IJIJ\nabla_{e_i}\theta\rangle = -\delta(JIJI\theta) - (2n-3)\langle IJ\theta, JI\theta\rangle + \mathrm{tr}(IJ)\langle I\theta, J\theta\rangle + |\theta|^2.$$

Substituting these in (4.40), we obtain

$$\begin{aligned}(4n^2 - 10n + 4)|\theta|^2 - (4n-8)\mathrm{tr}(IJ)\langle I\theta, J\theta\rangle - (4n-12)\langle IJ\theta, JI\theta\rangle \\ - \mathrm{tr}(IJIJ)|\theta|^2 + (\mathrm{tr}(IJ))^2|\theta|^2 = -(4n-6)\delta\theta - 2\delta(\mathrm{tr}(IJ)JI\theta) + 2\delta(JIJI\theta).\end{aligned} \tag{4.41}$$

In order to exploit this formula we need to distinguish two cases.

Case 1: If $n = 2$, (4.41) becomes:

$$4\langle IJ\theta, JI\theta\rangle - \mathrm{tr}(IJIJ)|\theta|^2 + (\mathrm{tr}(IJ))^2|\theta|^2 = -2\delta(\theta + \mathrm{tr}(IJ)JI\theta - JIJI\theta). \tag{4.42}$$

We claim that I and J define opposite orientations on $\mathrm{T}M$. Assume for a contradiction that they define the same orientation, and recall that complex structures compatible with the orientation on an oriented 4-dimensional Euclidean vector space may be identified with imaginary quaternions of norm 1 acting on $\mathbb{H}$ by left multiplication. For any $q, v \in \mathbb{H}$, we have: $\langle qv, v\rangle = \frac{1}{4}\mathrm{tr}(q)|v|^2$, where $\mathrm{tr}(q)$ denotes the trace of q acting by left multiplication on $\mathbb{H}$. For every $x \in M$ we can identify T_xM with $\mathbb{H}$ and view I, J as unit quaternions acting by left multiplication. The previous relation gives the following pointwise equality: $4\langle IJIJ\theta, \theta\rangle = \mathrm{tr}(IJIJ)|\theta|^2$. Substituting in (4.42) and integrating over M, implies that $\mathrm{tr}(IJ)=0$, so I and J anti-commute. Equation (4.42) then further implies that $\delta\theta=0$. Replacing these two last equalities in (4.39) yields $\sum_{i=1}^{2n}\langle e_i, J\nabla_{Ie_i}\theta\rangle IY-|\theta|^2 JY=0$. Since $\nabla\theta$ is symmetric and IJ is skew-symmetric, the first term vanishes, showing that $\theta\equiv 0$. This contradicts the assumption that the lcK structure (g,J,θ) is proper. Hence, I and J define opposite orientations and thus they commute.

Since (M,g,I) is a compact Kähler manifold, it follows that its first Betti number is even. N. Buchdahl [7] and A. Lamari [15] proved that each compact complex surface with even first Betti number carries a Kähler metric. On the other hand, I. Vaisman proved that if a complex manifold (M,J) admits a J-compatible Kähler metric, then every lcK metric on (M,J) is gcK [23, Theorem 2.1]. This shows that θ is exact.

Case 2. We assume from now on that $n\geq 3$. The integral over the compact manifold M of the left hand side of (4.41) is zero, since the right hand side is the co-differential of a 1-form. On the other hand, the following inequalities hold:

$$-(4n-8)\mathrm{tr}(IJ)\langle I\theta, J\theta\rangle\geq-(4n-8)|\mathrm{tr}(IJ)||\theta|^2\geq-\left((\mathrm{tr}(IJ))^2+(2n-4)^2\right)|\theta|^2, \tag{4.43}$$

with equality if and only if $I\theta=\pm J\theta$ and $\mathrm{tr}(IJ)=\pm(2n-4)$,

$$-(4n-12)\langle IJ\theta, JI\theta\rangle\geq-(4n-12)|\theta|^2, \tag{4.44}$$

(it is here that the assumption $n\geq 3$ is needed), and

$$-\mathrm{tr}(IJIJ)\geq-2n, \tag{4.45}$$

with equality if and only if $(IJ)^2=\mathrm{Id}$, which is equivalent to $IJ=JI$. Summing up the inequalities (4.43)–(4.45) shows that the left hand side of (4.41) is non-negative, hence it vanishes. Consequently, equality holds in (4.43)–(4.45), so I and J commute and (after replacing I with $-I$ if necessary) $I\theta=J\theta$ and $\mathrm{tr}(IJ)=2(n-2)$. It follows that the orthogonal involution IJ has two eigenvalues: 1 with multiplicity $2(n-1)$ and -1 with multiplicity 2. Let M' denote the set of points where θ is not zero. At each point of M', since the vectors θ and $I\theta$ are eigenvectors of IJ for the eigenvalue -1, it follows that $IJX=X$, for every X orthogonal on θ and $J\theta$, which can also be written as

$$JX=-IX+\frac{2}{|\theta^2|}\left(\langle X,\theta\rangle I\theta-\langle X, I\theta\rangle\theta\right),\qquad\forall X\in\mathrm{T}M'. \tag{4.46}$$

We thus have $\Omega^J=-\Omega^I+\frac{2}{|\theta^2|}\theta\wedge I\theta$ on M'. In particular, we have

$$\theta\wedge\Omega^J=-\theta\wedge\Omega^I, \tag{4.47}$$

at every point of M (as this relation holds tautologically on $M\setminus M'$, where by definition $\theta=0$). From (4.2) and (4.47) we get

$$\mathrm{d}\Omega^J=2\theta\wedge\Omega^J=-2\theta\wedge\Omega^I=-2L(\theta), \tag{4.48}$$

where $L:\Lambda^*M\to\Lambda^*M$, $L(\alpha):=\Omega^I\wedge\alpha$ is the Lefschetz operator defined on the Kähler manifold (M,g,I).

Using the Hodge decomposition on M, we decompose the closed 1-form θ as $\theta = \theta_H + \mathrm{d}\varphi$, where θ_H is the harmonic part of θ and φ is a smooth real-valued function on M. From (4.48) and the fact that L commutes with the exterior differential, we obtain

$$L(\theta_H) = -\mathrm{d}\left(\frac{1}{2}\Omega^J + L\varphi\right). \tag{4.49}$$

Moreover, since L commutes on any Kähler manifold with the Laplace operator (see *e.g.* [18]), the left-hand side of (4.49) is a harmonic form and the right-hand side is exact. This implies that $L\theta_H$ vanishes, so $\theta_H = 0$ since L is injective on 1-forms for $n \geq 2$. Thus $\theta = \mathrm{d}\varphi$ is exact, so (M, g, J, θ) is globally conformally Kähler.

Example 4.18 *As in Example 4.7, we consider on $M := \mathbb{C}^n \setminus \{0\}$ the standard flat structure (g_0, J_0). Let J be a constant complex structure on M, compatible with g_0 and which does not commute with J_0. Then, $(M, g := r^{-4}g_0, J)$ is gcK and (M, g, I) is Kähler, where I is the pull-back of J_0 through the inversion, but J and I do not commute. This example shows that the compactness assumption in Theorem 4.17 is necessary.*

6 Conformal classes with non-homothetic Kähler metrics

As an application of Theorem 4.17, we will describe in this section all compact conformal manifolds (M^{2n}, c) with $n \geq 2$, such that the conformal class c contains two non-homothetic Kähler metrics.

We start by constructing a class of examples, which will be referred to as the Calabi Ansatz.

Example 4.19 (Calabi Ansatz) *Let (N, h, J_N, Ω_N) be a Hodge manifold,* i.e. *a compact Kähler manifold with $[\Omega_N] \in H^2(N, 2\pi\mathbb{Z})$. Let $\pi : S \to N$ be the principal S^1-bundle with the connection (given by Chern-Weil theory) whose curvature form is the pull-back to S of $i\Omega_N$. For any positive real number ℓ, let h_ℓ be the unique Riemannian metric on S such that π is a Riemannian submersion with fibers of length $2\pi\ell$. Then for every $b > 0$ and smooth function $\ell : (0, b) \to \mathbb{R}^{>0}$, the metric $g_\ell := h_{\ell(r)} + \mathrm{d}r^2$ on $M' := S \times (0, b)$ is globally conformally Kähler with respect to two distinct complex structures. Moreover, if $\ell^2(r) = r^2(1 + A(r^2))$ near $r = 0$ and $\ell^2(r) = (b - r)^2(1 + B((b - r)^2))$ near $r = b$, for smooth functions A, B defined near 0 with $A(0) = B(0) = 0$, then g_ℓ extends smoothly to a Riemannian metric g_0 on the metric completion M of (M', g_ℓ), which is an S^2-bundle over N.*

Proposition 4.20 *Let (N, h, J_N, Ω_N) be a Hodge manifold, $b \in \mathbb{R}^{>0}$ and $\ell : (0, b) \to \mathbb{R}^{>0}$ a smooth function satisfying the boundary conditions as above. Then, the metric g_0 constructed in Example 4.19 on the total space M of the S^2-bundle over N is globally conformally Kähler with respect to two distinct complex structures on M.*

Proof. Let $i\omega \in \Omega^1(S, i\mathbb{R})$ denote the connection form on S satisfying

$$\mathrm{d}\omega = \pi^*(\Omega_N). \tag{4.50}$$

The metric h_ℓ is defined by
$$h_\ell := \pi^* h + \ell^2 \omega \otimes \omega.$$
Let ξ denote the vector field on S induced by the S^1-action. By definition ξ verifies $\pi_*\xi = 0$ and $\omega(\xi) = 1$. Let X^* denote the horizontal lift of a vector field X on N (defined by $\omega(X^*) = 0$ and $\pi_*(X^*) = X$). By the equivariance of the connection we have $[\xi, X^*] = 0$ for every vector field X, and from (4.50) we readily obtain $[X^*, Y^*] = [X, Y]^* - \Omega_N(X, Y)\xi$. The Koszul formula immediately gives the covariant derivative ∇^ℓ of the metric $g_\ell := h_{\ell(r)} + \mathrm{d}r^2$ on $M' := S \times (0, b)$:

$$\begin{aligned}
\nabla^\ell_\xi \partial_r &= \nabla^\ell_{\partial_r}\xi = \frac{\ell'}{\ell}\xi,\\
\nabla^\ell_\xi \xi &= -\ell\ell'\partial_r,\\
\nabla^\ell_{\partial_r}\partial_r &= \nabla^\ell_{X^*}\partial_r = \nabla^\ell_{\partial_r}X^* = 0,\\
\nabla^\ell_{X^*}\xi &= \nabla^\ell_\xi X^* = \frac{\ell^2}{2}(J_N X)^*,\\
\nabla^\ell_{X^*}Y^* &= (\nabla^h_X Y)^* - \frac{1}{2}\Omega_N(X, Y)\xi.
\end{aligned}$$

We now define for $\varepsilon = \pm 1$ the Hermitian structures J_ε on (M', g_ℓ) by
$$J_\varepsilon(X^*) := \varepsilon(J_N X)^*, \qquad J_\varepsilon(\xi) := \ell\partial_r, \qquad J_\varepsilon(\partial_r) := -\ell^{-1}\xi.$$
A straightforward calculation using the previous formulas yields $\nabla^\ell_U J_\varepsilon = Z \wedge J_\varepsilon\theta_\varepsilon + J_\varepsilon Z \wedge \theta_\varepsilon$ for every vector field Z on M, where $\theta_\varepsilon := \frac{1}{2}\varepsilon\ell\mathrm{d}r$. Thus (g_ℓ, J_ε) are globally conformally Kähler structures on M' with Lee forms
$$\theta_\varepsilon = \varepsilon\mathrm{d}\varphi, \qquad \text{where} \qquad \varphi(r) := \frac{1}{2}\int_0^r \ell(t)\mathrm{d}t.$$

The last statement follows immediately from a coordinate change (from polar to Euclidean coordinates) in the fibers $S^1 \times (0, b)$ of the Riemannian submersion $M' \to N$. Indeed, in a neighbourhood of $r = 0$, with Euclidean coodinates $x_1 := r\cos t$ and $x_2 := r\sin t$, we have:
$$\begin{pmatrix}\partial_r\\ \frac{1}{r}\xi\end{pmatrix} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix},$$
where $\xi = \partial_t$. In these coordinates, we have the following formulas for the complex structures and the metric:
$$\begin{aligned}
J_\varepsilon(\partial_1) &= \frac{\ell}{r}\left(\frac{-A(r^2)}{\ell^2}x_1x_2\partial_1 - \left(\frac{A(r^2)}{\ell^2}x_1^2 + 1\right)\partial_2\right),\\
J_\varepsilon(\partial_2) &= \frac{\ell}{r}\left(\frac{A(r^2)}{\ell^2}x_1x_2\partial_2 + \left(1 - \frac{A(r^2)}{\ell^2}x_2^2\right)\partial_1\right),\\
g = \pi^*h + \left(1 + \frac{A(r^2)}{r^2}x_2^2\right)&\mathrm{d}x_1^2 + \left(1 + \frac{A(r^2)}{r^2}x_1^2\right)\mathrm{d}x_2^2 - 2\frac{A(r^2)}{r^2}x_1x_2\mathrm{d}x_1\mathrm{d}x_2.
\end{aligned}$$

From the assumption on A, the functions $\frac{\ell}{r}$ and $\frac{A(r^2)}{\ell^2}$ extend smoothly at $r=0$, therefore, the complex structures J_-, J_+ and the metric g extend smoothly at $r=0$. The same argument applies to the other extremal point $r=b$. Hence, the metric g_ℓ on M' extends to a smooth metric g_0 on M, and there exist two distinct Kähler structures on M in the conformal class $[g_0]$, whose restrictions to M' are equal to $(g_+ := e^{\varphi} g_\ell, J_+)$ and $(g_- := e^{-\varphi} g_\ell, J_-)$ respectively.

Conversely, we have the following:

Proposition 4.21 *Let (M, g_0, I) be a compact globally conformally Kähler manifold with non-trivial Lee form $\theta_0 = d\varphi_0$. We assume that on M', the set where θ_0 is not vanishing, its derivative has the following form:*

$$\nabla^0_X \theta_0 = f(\theta_0(X)\theta_0 + I\theta_0(X) I\theta_0), \quad \forall X \in \mathrm{T}M', \tag{4.51}$$

where $f \in C^\infty(M')$ and ∇^0 is the Levi-Civita connection of g_0. We further assume that there exists a distribution $\mathcal{V}$ on M, which on M' is spanned by ξ and $I\xi$, where ξ is the metric dual of $I\theta_0$ with respect to g_0. Then (M, g_0, I) is obtained from the Calabi Ansatz described in Example 4.19.

Proof. We first notice that $M' \neq M$. Indeed, θ_0 vanishes at the extrema of the function φ_0 defined on the compact manifold M.

From (4.51) and (4.1) we deduce the following formulas on M':

$$\nabla^0_X (I\xi) = -f(\langle X, I\xi \rangle I\xi + \langle X, \xi \rangle \xi), \quad \forall X \in \mathrm{T}M', \tag{4.52}$$

$$\nabla^0_X \xi = (1+f)(\langle X, \xi \rangle I\xi - \langle X, I\xi \rangle \xi) - |\xi|^2 IX, \quad \forall X \in \mathrm{T}M', \tag{4.53}$$

which imply that the distribution $\mathcal{V}$ is totally geodesic.

Equation (4.53) also shows that $\nabla^0 \xi$ is a skew-symmetric endomorphism, hence ξ is a Killing vector field with respect to the metric g_0. We denote by N one of the connected components of the zero set of the Killing vector field ξ, which is then a compact totally geodesic submanifold of M. Since $\mathrm{d}(I\theta_0)$ has rank 2 along N, it follows that N has co-dimension 2.

Let Φ_s denote the 1-parameter group of isometries of (M, g_0) induced by ξ and let us fix some $p \in N$. For every $s \in \mathbb{R}$, the differential of Φ_s at p is an isometry of T_pM which fixes T_pN, so it is determined by a rotation of angle $k(s)$ in $\mathcal{V}_p$. From $\Phi_s \circ \Phi_{s'} = \Phi_{s+s'}$ we obtain $k(s) = ks$, for some $k \in \mathbb{R}^*$. For $s_0 = 2\pi/k$, the isometry Φ_{s_0} fixes p and its differential at p is the identity. We obtain that $\Phi_{s_0} = \mathrm{Id}_M$, so ξ has closed orbits. Note that any $p \in N$ is a fixed point of Φ_s, for all $s \in \mathbb{R}$, and for $s = \frac{s_0}{2}$, $\Phi_{\frac{s_0}{2}}$ is an orientation preserving isometry whose differential at p squares to the identity, and is the identity on $\mathrm{T}_pN = \mathcal{V}_p^\perp$. Hence, $(\mathrm{d}\Phi_{\frac{s_0}{2}})_p|_{\mathcal{V}_p}$ is either plus or minus the identity of $\mathcal{V}_p$. The first possibility would contradict the definition of s_0, so we have

$$(\mathrm{d}\Phi_{\frac{s_0}{2}})_p|_{\mathcal{V}_p} = -\mathrm{Id}_{\mathcal{V}_p}. \tag{4.54}$$

Let γ be a geodesic of (M, g_0) starting from p, such that $V := \dot\gamma(0) \in \mathcal{V}_p$ and $|\dot\gamma(0)| = 1$. Since $\mathcal{V}$ is totally geodesic, $\dot\gamma(t) \in \mathcal{V}$ for all t. The derivative of the function

$g_0(\xi, \dot\gamma)$ along γ equals $g_0(\nabla^0_{\dot\gamma}\xi, \dot\gamma) = 0$ and the function clearly vanishes at $t = 0$, so $g(\xi, \dot\gamma) \equiv 0$ along γ. We thus have

$$I\xi_{\gamma(t)} = c_{p,V}(t)\dot\gamma(t), \tag{4.55}$$

for some function $c_{p,V} : \mathbb{R} \to \mathbb{R}$. Clearly $c^2_{p,V}(t) = |\xi_{\gamma(t)}|^2$, so $c_{p,V}$ is smooth at all points t with $\gamma(t) \in M'$. By (4.52)–(4.53) we easily check that $[\xi, I\xi] = 0$. Hence, each isometry Φ_s preserves $I\xi$. Moreover, $\Phi_s(\gamma(t))$ is the geodesic starting at p with tangent vector $(\Phi_s)_*(\dot\gamma(0))$. We thus see that the function $c_p := c_{p,V}$ does not depend on the unit vector V in $\mathcal{V}_p$ defining γ.

We claim that in fact, for all $p, q \in N$, $c_p(t) = c_q(t)$, for all t. In other words, the norm of $\xi_{\gamma(t)}$ only depends on t and not on the initial data of γ starting in N. For a fixed $t \in \mathbb{R}$, we consider the map $F : SN \to M$, $F(V) := \exp(tV)$, where SN denotes the unit normal bundle of N. By the Gauss' Lemma, we know that $\mathrm{d}F_V(T_V SN) \subset (\dot\gamma_{p,V}(t))^\perp$, where $\gamma_{p,V}$ denotes the geodesic starting at p with unit speed vector V. Since ξ is a Killing vector field, the function $g_0(\dot\gamma_{p,V}, \xi)$ is constant along $\gamma_{p,V}$ and thus identically zero, because ξ vanishes on N. As $\dot\gamma_{p,V} \in \mathcal{V}$, it follows that $\dot\gamma_{p,V}$ is proportional to $I\xi$, which is the metric dual of $-\theta_0$. On the other hand, we know that $\mathrm{d}|\theta_0|^2 = 2f|\theta_0|^2\theta_0$. Therefore, $\mathrm{d}|\theta_0|^2$ vanishes on $\mathrm{d}F_V(T_V SN)$, showing that the norm of $\xi_{\gamma(t)}$ does not depend on the starting point either. Hence, we further denote the function $c_p = c_{p,V}$ simply by $c : \mathbb{R} \to \mathbb{R}$.

Differentiating the relation $\gamma_{p,V}(t) = \gamma_{p,-V}(-t)$ which holds for all geodesics and for all t, yields $\dot\gamma_{p,V}(t) = -\dot\gamma_{p,-V}(-t)$. Therefore, from (4.55) we conclude that $c(-t) = -c(t)$, for all t. Moreover, $c(t)$ is non-vanishing for $|t| \neq 0$ and sufficiently small. By changing the orientation of M if necessary, we thus can assume that c is negative on some interval $(0, \varepsilon)$ and positive on $(-\varepsilon, 0)$. Since $(\varphi_0(\gamma(t)))' = \theta_0(\dot\gamma(t)) = -c(t)$, we conclude that N is a connected component of the level set of a local maximum of φ_0.

By compactness of N, the exponential map defined on the normal bundle of N is surjective, so its image contains points where φ_0 attains its absolute minimum. At such a point, the vector field ξ vanishes, so (4.55) shows that $t_0 := \inf\{t > 0 \,|\, c(t) = 0\}$ is well-defined and positive. Let N' be a connected component of the inverse image through φ_0 of $\varphi_0(\exp_p(t_0V))$, for some $p \in N$ and some unit vector V in $\mathcal{V}_p$. The above argument, applied to N' instead of N, shows that N' is a connected component of the level set of a local minimum of φ_0. It also shows that $\exp_q(t_0W) \in N'$ for any $q \in N$ and any unit vector $W \in \mathcal{V}_q$. From (4.54) it follows that $\dot\gamma_{p,-V}(t_0) = -\dot\gamma_{p,V}(t_0)$, for any $p \in N$ and any unit vector $V \in \mathcal{V}_p$. In other words, if a geodesic starting at a point p of N with unit speed vector $V \in \mathcal{V}_p$ arrives after time t_0 in a point $p' \in N'$ with speed vector $V' \in \mathcal{V}_{p'}$, then the geodesic starting at p with speed vector $-V$ arrives after time t_0 in p' with speed vector $-V'$, showing that these two geodesics close up to one geodesic. Hence, M equals the image through the exponential map of the compact subset of the normal bundle of N consisting of vectors of norm $\leq t_0$, thus showing that $M \setminus M' = N \cup N'$.

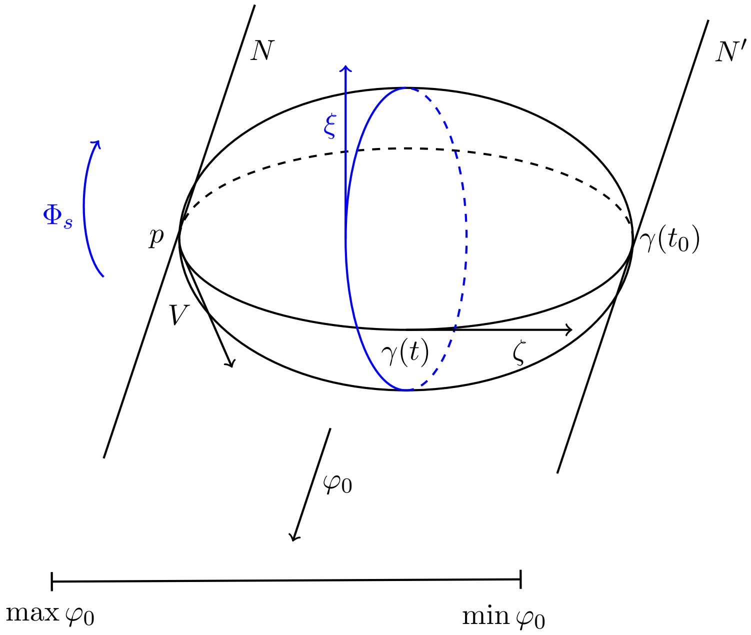

Figure 4.1: Visualization of the vector fields ξ and ζ

Consequently, the function φ_0 attains its maximum on N and its minimum on N' and has no other critical point. Let S be some level set corresponding to a regular value of φ_0. Consider the unit vector field $\zeta := \frac{I\xi}{|I\xi|}$ on M'. See Figure 4.1 for a visualization of the vector fields ξ and ζ and of the level sets of φ_0.

From (4.52) we readily compute

$$\nabla^0_X \zeta = -\frac{f}{|\xi|}\langle X, \xi \rangle \xi. \tag{4.56}$$

In particular, we have $\nabla^0_\zeta \zeta = 0$. So, if Ψ denotes the (local) flow of ζ, the curve $t \mapsto \Psi_t(x)$ is a geodesic for every $x \in M'$, that is, $\Psi_t(x) = \exp_x(t\zeta)$. Note that by (4.56), we have $\mathrm{d}\zeta^\flat = 0$ so the Cartan formula implies $\mathcal{L}_\zeta \zeta^\flat = \mathrm{d}(\zeta \lrcorner \zeta^\flat) + \zeta \lrcorner \mathrm{d}\zeta^\flat = 0$, which can also be written as

$$(\mathcal{L}_\zeta g_0)(\zeta, X) = 0, \qquad \forall X \in \mathrm{T}M'. \tag{4.57}$$

We claim that for fixed t, $\varphi_0(\Psi_t(x))$ does not depend on $x \in S$. To see this, let $X \in \mathrm{T}_x S$. By definition $\mathrm{d}\varphi_0(X) = 0$, whence $g_0(X, \zeta) = 0$. We need to show that $\mathrm{d}\varphi_0((\Psi_t)_*(X)) = 0$. This is equivalent to $0 = g_0(\zeta, (\Psi_t)_*(X)) = (\Psi_t^* g_0)(\zeta, X)$, which clearly holds at $t = 0$. Moreover, from (4.57) we see that the derivative of the function $(\Psi_t^* g_0)(\zeta, X)$ vanishes:

$$\frac{\mathrm{d}}{\mathrm{d}t}((\Psi_t^* g_0)(\zeta, X)) = (\Psi_t^* \mathcal{L}_\zeta g_0)(\zeta, X) = (\mathcal{L}_\zeta g_0)(\zeta, (\Psi_t)_*(X)) = 0.$$

This shows that for every $x \in S$, $\exp_x(t\zeta)$ belongs to the same level set of φ_0. Take the smallest $t_1 > 0$ such that $\pi(x) := \exp_x(t_1\zeta) \in N$ for every $x \in S$.

Claim. The map π is a Riemannian submersion from $(S, g_0|_S)$ to $(N, g_0|_N)$ with totally geodesic fibers tangent to ξ.

Proof of the Claim. First, the Killing vector field ξ commutes with ζ, so $(\Psi_t)_*\xi = \xi$ for all $t < t_1$. Making t tend to t_1 implies $\pi_*(\xi_x) = \xi_{\pi(x)} = 0$ for every $x \in S$, since $\pi(x) \in N$. Thus $I\zeta$ is tangent to the fibers of π. From (4.53) we get $\nabla_{I\zeta} I\zeta = f|\xi|\zeta$, so $I\zeta$ is a geodesic vector field on S.

Take now any tangent vector $X \in \mathrm{T}_x S$ orthogonal to $I\zeta$ and denote by $X_t := (\Psi_t)_*(X)$, which makes sense for all $t < t_1$. By construction we have $\pi_*(X) = \lim_{t \to t_1} X_t$. Since $0 = [\zeta, X_t] = \nabla^0_\zeta X_t - \nabla^0_{X_t}\zeta$, we get by (4.56)

$$\zeta(\langle X_t, I\zeta \rangle) = \langle \nabla^0_\zeta X_t, I\zeta \rangle + \langle X_t, \nabla^0_\zeta I\zeta \rangle = \langle \nabla^0_{X_t}\zeta, I\zeta \rangle = -f|\xi|\langle X_t, I\zeta \rangle.$$

The function $\langle X_t, I\zeta \rangle$ vanishes at $t = 0$ and satisfies a first order linear ODE along the geodesic $\gamma(t) := \exp_x(t\zeta)$, so it vanishes identically. Thus, X_t is orthogonal to $I\zeta$ for all $t < t_1$. Moreover, the vector field X_t along γ has constant norm:

$$\zeta(|X_t|^2) = 2\langle \nabla^0_\zeta X_t, X_t \rangle = 2\langle \nabla^0_{X_t}\zeta, X_t \rangle \overset{(4.56)}{=} -\frac{2f}{|\xi|}\langle X_t, \xi \rangle = 0. \tag{4.58}$$

This shows that $|\pi_*(X)|^2 = |X|^2$, thus proving the claim.

Let us now consider the smallest $t_2 > 0$ such that $\pi(x) := \exp_x(-t_2\zeta) \in N'$ for every $x \in S$ and let $b := t_1 + t_2$. The flow of the geodesic vector field ζ defines a diffeomorphism between $(0, b) \times S$ and M', which maps (r, x) onto $\exp_x((r - t_2)\zeta)$. With respect to this diffeomorphism, the vector field ζ is equal to $\zeta = \partial_r$, the metric reads $g_0 = \mathrm{d}r^2 + k_r$, where k_r is a family of Riemannian metrics on S and the function $|\theta_0|$ only depends on r, say $|\theta_0| = \alpha(r)$. It follows that $\theta_0 = \alpha \mathrm{d}r$ and since $\mathrm{d}\varphi_0 = \theta_0$, we see that $\varphi_0 = \varphi_0(r)$ and

$$\varphi_0' = \alpha. \tag{4.59}$$

The previous claim actually shows that for every $r \in (0, b)$,

$$k_r = \pi^*(h) + \tau_r \otimes \tau_r,$$

where $\tau_r := I\zeta^\flat$ and $h := g_0|_N$. From (4.56) we readily obtain

$$\dot{\tau}_r = \mathcal{L}_\zeta(I\zeta^b) = -f\alpha I\zeta^b = -f\alpha\tau_r.$$

This shows that $\tau_r = \ell(r)\omega$ with $\ell(r) := e^{-\int_0^r f(t)\alpha(t)\mathrm{d}t}$, where ω denotes the connection 1-form on the S^1-bundle $S \to N$ induced by the Riemannian submersion π. Finally, the metric on M' reads $g_0 = \mathrm{d}r^2 + \pi^*(h) + \ell^2\omega \otimes \omega$, showing that g_0 has the form of the metric described in Example 4.19.

6.1 Proof of Theorem 4.2

We can now finish the classification of compact manifolds carrying two conformally related non-homothetic Kähler metrics. Assume that (g_+, J_+) and (g_-, J_-) are Kähler structures on a compact manifold M of real dimension $2n \geq 4$ with $g_+ = e^{2\varphi} g_-$ for some non-constant function φ. Note that J_+ is not conjugate to J_-. Indeed, if J_+ were equal to $\pm J_-$, then $\Omega_+ = \pm e^{2\varphi}\Omega_-$, so $0 = \mathrm{d}\Omega_+ = \pm 2e^{2\varphi}\mathrm{d}\varphi \wedge \Omega_-$ would imply $\mathrm{d}\varphi = 0$, so φ would be constant.

In order to use the results from Section 5, we make the following notation:

$$g := g_+, \quad I := J_+, \quad J := J_-, \quad \Omega^I := \Omega_+ = g(J_+\cdot, \cdot), \quad \Omega^J := g(J\cdot, \cdot) = e^{2\varphi}\Omega_-.$$

Then (M, g, I) is Kähler, and (M, g, J) is lcK (in fact globally conformally Kähler), with Lee form $\theta := \mathrm{d}\varphi$. This last statement follows from (4.2), since $\mathrm{d}\Omega^J = 2e^{2\varphi}\mathrm{d}\varphi \wedge \Omega_- = 2\mathrm{d}\varphi \wedge \Omega^J$.

The first part of Theorem 4.17 shows that I and J commute, which proves the statment of Theorem 4.2 for $n = 2$. For $n \geq 3$, the proof of the Theorem 4.17 shows moreover, that after replacing I by $-I$ if necessary, one has $I\theta = J\theta$ and $\mathrm{tr}(IJ) = 2(n-2)$.

Let us now consider the 2-form $\sigma := \frac{1}{2}\Omega^I + \frac{1}{2}\Omega^J$, corresponding to the endomorphism $I + J$ of $\mathrm{T}M$ via the metric g. By (4.46), on M' we have

$$\sigma = \frac{1}{|\theta|^2}\theta \wedge I\theta. \tag{4.60}$$

Since I is ∇-parallel (where ∇ is the Levi-Civita connection of g), we obtain by (4.1) that $\nabla_X \sigma = \frac{1}{2}(X \wedge J\theta + JX \wedge \theta)$. Substituting $J = 2\sigma - I$, and using the fact that $\sigma(\theta) = I\theta$ we obtain the following formula for the covariant derivative of σ:

$$\begin{aligned} \nabla_X \sigma &= \frac{1}{2}\nabla_X J = \frac{1}{2}(X \wedge J\theta + JX \wedge \theta) \\ &= \frac{1}{2}(X \wedge (2\sigma - I)\theta + (2\sigma - I)X \wedge \theta) \\ &= \frac{1}{2}(X \wedge I\theta - IX \wedge \theta) + \sigma(X) \wedge \theta. \end{aligned}$$

Since by (4.60) $\theta \wedge \sigma = 0$, we get $0 = X \lrcorner (\theta \wedge \sigma) = \langle X, \theta \rangle \sigma - \theta \wedge \sigma(X)$ for every $X \in \mathrm{T}M$. The previous computation thus yields

$$\nabla_X \sigma = \frac{1}{2}(X \wedge I\theta - IX \wedge \theta) - \langle X, \theta \rangle \sigma, \qquad \forall X \in \mathrm{T}M. \tag{4.61}$$

We consider now the 2-form $\widetilde{\sigma} := e^{\varphi}\sigma$, whose covariant differential is computed by (4.61) as follows:

$$\nabla_X \widetilde{\sigma} = \frac{e^{\varphi}}{2}(X \wedge I\theta - IX \wedge \theta), \quad \forall X \in \mathrm{T}M.$$

Equivalently, this equation can be written as

$$\nabla_X \widetilde{\sigma} = \frac{1}{2}(\mathrm{d}(\mathrm{tr}\, \widetilde{\sigma}) \wedge IX - \mathrm{d}^c(\mathrm{tr}\, \widetilde{\sigma}) \wedge X), \quad \forall X \in \mathrm{T}M, \tag{4.62}$$

where $\mathrm{tr}\, \widetilde{\sigma} := \langle \widetilde{\sigma}, \Omega^I \rangle = e^{\varphi}$ is the trace with respect to the Kähler form Ω^I and d^c denotes the twisted exterior differential defined by $\mathrm{d}^c\alpha := \sum_i Ie_i \wedge \nabla_{e_i}\alpha$, for any form α.

A real $(1,1)$-form on a Kähler manifold (M, g, I, Ω^I) satisfying (4.62) is called a *Hamiltonian 2-form* (see [1]). Compact Kähler manifolds carrying such forms are completely described in [3, Theorem 5]. In the case where the form is decomposable, these are exactly as in Example 4.19.

However, the statement and the proof of [3, Theorem 5] are rather involved, and it is not completely clear that the construction described in the conclusion of [3, Theorem 5] is equivalent to the one from Example 4.19. We will thus provide here a more direct proof in our particular case.

All we need is to show that the globally conformally Kähler structure on M determined by $g_0 := e^{\varphi} g_- = e^{-\varphi} g_+$ and $J := J_-$ satisfies the hypotheses of Proposition 4.21. We start with the following:

Lemma 4.22 *On the open set M' where θ is not vanishing, the covariant derivative of θ with respect to g is given by*

$$\nabla_X \theta = \frac{1}{2}|\theta|^2 X - \frac{1}{2}\left(\frac{\delta\theta}{|\theta|^2} + n + 1\right)\langle X, \theta\rangle\theta - \frac{1}{2}\left(\frac{\delta\theta}{|\theta|^2} + n - 1\right)\langle X, I\theta\rangle I\theta. \quad (4.63)$$

Proof. Using the fact that I and J commute and $\mathrm{tr}(IJ) = 2(n-2)$, (4.40) simplifies to

$$\sum_{i=1}^{2n}\langle IJ\nabla_{e_i}\theta, e_i\rangle = 2(n-1)|\theta|^2 + \delta\theta. \quad (4.64)$$

Substituting this into (4.39), we obtain

$$\begin{aligned}2(2-n)\langle Y, \theta\rangle J\theta - 2n\langle Y, J\theta\rangle\theta + (2n-5)|\theta|^2 JY + |\theta|^2 IY + \delta\theta(JY + IY)\\ + 2(2-n)J\nabla_Y\theta + 2\nabla_{JY}\theta + 2(n-2)\nabla_{IY}\theta - 2IJ\nabla_{IY}\theta = 0. \quad (4.65)\end{aligned}$$

Differentiating (4.60) on M' yields

$$\nabla_X\sigma = -\frac{2\langle\nabla_X\theta, \theta\rangle}{|\theta|^4}\theta\wedge I\theta + \frac{1}{|\theta|^2}(\nabla_X\theta\wedge I\theta + \theta\wedge I\nabla_X\theta). \quad (4.66)$$

Comparing with (4.61), we obtain

$$\frac{1}{2}(X\wedge I\theta - IX\wedge\theta) - \langle X, \theta\rangle\sigma = -\frac{2\langle\nabla_X\theta, \theta\rangle}{|\theta|^4}\theta\wedge I\theta + \frac{1}{|\theta|^2}(\nabla_X\theta\wedge I\theta + \theta\wedge I\nabla_X\theta). \quad (4.67)$$

Taking the interior product with $I\theta$ in the last equality, we get

$$\frac{1}{2}\langle X, I\theta\rangle I\theta - \frac{1}{2}|\theta|^2 X + \frac{1}{2}\langle X, \theta\rangle\theta = \frac{\langle\nabla_X\theta, \theta\rangle}{|\theta|^2}\theta + \frac{1}{|\theta|^2}\langle\nabla_X\theta, I\theta\rangle I\theta - \nabla_X\theta. \quad (4.68)$$

We deduce that the following equality holds:

$$\nabla_X\theta = \frac{1}{2}|\theta|^2 X + \alpha(X)\theta + \beta(X)I\theta, \quad (4.69)$$

where α and β are the following 1-forms:

$$\alpha = \frac{1}{2}\left(\frac{\mathrm{d}(|\theta|^2)}{|\theta|^2} - \theta\right), \quad \beta = \frac{1}{|\theta|^2}\nabla_{I\theta}\theta - \frac{1}{2}I\theta. \tag{4.70}$$

Since θ is closed, (4.69) yields

$$0 = \alpha \wedge \theta + \beta \wedge I\theta.$$

Therefore, there exist $a, b, c \in C^\infty(M')$, such that

$$\alpha = a\theta + bI\theta \text{ and } \beta = b\theta + cI\theta.$$

Moreover, (4.70) shows that α is closed, so $\mathrm{d}a \wedge \theta + \mathrm{d}b \wedge I\theta + b\mathrm{d}(I\theta) = 0$. On the other hand, by (4.69), we have $\mathrm{d}(I\theta) = |\theta|^2\Omega^I + \alpha \wedge I\theta - \beta \wedge \theta = |\theta|^2\Omega^I + (a+c)\theta \wedge I\theta$. Hence,

$$\mathrm{d}a \wedge \theta + \mathrm{d}b \wedge I\theta + b|\theta|^2\Omega^I + b(a+c)\theta \wedge I\theta = 0.$$

Applying the last equality to X and IX, for a non-zero vector field X orthogonal to θ and $I\theta$ yields $b = 0$. By (4.69) again we have

$$-\delta\theta = \sum_{i=1}^{2n}\langle\, e_i,\, \nabla_{e_i}\theta\,\rangle = (n+a+c)|\theta|^2. \tag{4.71}$$

Substituting Y by θ in (4.65) and using (4.69), we obtain

$$\left(\delta\theta + (1 + (2-n)a + nc)|\theta|^2\right) I\theta = 0. \tag{4.72}$$

From (4.71) and (4.72), it follows that

$$a = -\frac{1}{2}\left(\frac{\delta\theta}{|\theta|^2} + n + 1\right) \quad \text{and} \quad c = -\frac{1}{2}\left(\frac{\delta\theta}{|\theta|^2} + n - 1\right).$$

This proves the lemma.

We write (4.63) as

$$\nabla_X\theta = \frac{1}{2}g(\theta,\theta)X^\flat + \frac{1}{2}(f-2)\theta(X)\theta + \frac{1}{2}fI\theta(X)I\theta, \tag{4.73}$$

where $f := -\left(\frac{\delta\theta}{|\theta|^2} + n - 1\right)$. Note that we no longer identify vectors and 1-forms in this relation, since we will now perform a conformal change of the metric.

Namely, we consider the "average metric" $g_0 := e^\varphi g_- = e^{-\varphi}g_+$ and denote by ∇^0 its Levi-Civita covariant derivative, by θ_0 the Lee form of $J := J_-$ with respect to g_0 and by $\Omega_0 := g_0(J\cdot,\cdot)$. Since $\mathrm{d}\Omega_0 = \mathrm{d}(e^\varphi\Omega_-) = e^\varphi\mathrm{d}\varphi \wedge \Omega_- = \mathrm{d}\varphi \wedge \Omega_0$, we get $\theta_0 = \frac{1}{2}\mathrm{d}\varphi = \frac{1}{2}\theta$.

From (4.73) we immediately get

$$\nabla_X\theta_0 = g(\theta_0,\theta_0)X^\flat + (f-2)\theta_0(X)\theta_0 + fI\theta_0(X)I\theta_0. \tag{4.74}$$

The classical formula relating the covariant derivatives of g and g_0 on 1-forms reads

$$\nabla^0_X \eta = \nabla_X \eta - g(\theta_0, \eta) X^\flat + \eta(X)\theta_0 + \theta_0(X)\eta, \qquad \forall X \in \mathrm{T}M,\ \forall \eta \in \Omega^1(M),$$

where $\flat$ is the index lowering with respect to g. For $\eta = \theta_0$, (4.74) becomes exactly (4.51).

From the proof of Theorem 4.17 it is clear that the distribution $\mathcal{V} := \ker(I - J)$ is spanned along M' by ξ and $I\xi$, where ξ denotes the vector field on M corresponding to $I\theta_0$ via the metric g_0. This shows that the hypotheses of Proposition 4.21 are verified, thus concluding the proof of Theorem 4.2.

6.2 Proof of Theorem 4.3

Using the above results, we can now complete the classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with non-generic holonomy, by reviewing the possible cases in the Berger-Simons holonomy theorem.

First, by Proposition 4.16, there exist no compact irreducible locally symmetric proper lcK manifolds.

In Section 4.3, we showed that if the restricted holonomy of (M, g) is in the Berger list, then necessarily $\mathrm{Hol}_0(M, g) = \mathrm{U}(n)$. After passing to a double covering if neccesary, there exists a complex structure I, such that (M, g, I) is Kähler. By Theorem 4.17, I and J commute and (M, g, J, θ) is gcK. In other words, there exist in the conformal class of g two non-homothetic Kähler metrics. We conclude then by Theorem 4.2 that (M, g, J, θ) falls in one of the cases 2.a) or 2.b) in Theorem 4.3.

Finally, if $\mathrm{Hol}_0(M, g)$ is reducible, then Theorem 4.11 shows that necessarily we have $\mathrm{Hol}_0(M, g) = \mathrm{SO}(2n-1)$, and Theorem 4.15 implies that (M^{2n}, g, J, θ) satisfies either case 1. or case 2.c) in Theorem 4.3.

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Chapter 5

Toric Vaisman Manifolds

Mihaela Pilca

Abstract. Vaisman manifolds are strongly related to Kähler and Sasaki geometry. In this paper we introduce toric Vaisman structures and show that this relationship still holds in the toric context. It is known that the so-called minimal covering of a Vaisman manifold is the Riemannian cone over a Sasaki manifold. We show that if a complete Vaisman manifold is toric, then the associated Sasaki manifold is also toric. Conversely, a toric complete Sasaki manifold, whose Kähler cone is equipped with an appropriate compatible action, gives rise to a toric Vaisman manifold. In the special case of a strongly regular compact Vaisman manifold, we show that it is toric if and only if the corresponding Kähler quotient is toric.

Keywords. Vaisman manifold, toric manifold, Sasaki structure, twisted Hamiltonian action, locally conformally Kähler manifold.

1 Introduction

Toric geometry has been studied intensively, as manifolds with many symmetries often occur in physics and also represent a large source of examples as testing ground for conjectures. The classical case of compact symplectic toric manifolds has been completely classified by T. Delzant [10], who showed that they are in one-to-one correspondence to the so-called Delzant polytopes, obtained as the image of the momentum map. Afterwards, similar classification results have been given in many different geometrical settings, some of which we briefly mention here. For instance, classification results were obtained by Y. Karshon and E. Lerman [19] for non-compact symplectic toric manifolds and by E. Lerman and S. Tolman [21] for symplectic orbifolds. The case when one additionally considers compatible metrics invariant under the toric action is also well understood: compact toric Kähler manifolds have been investigated by V. Guillemin [17], D. Calderbank, L. David and P. Gauduchon [8], M. Abreu [1], and compact toric Kähler orbifolds in [2]. Other more special structures have been completely classified, such as orthotoric Kähler, by V. Apostolov, D. Calderbank and P. Gauduchon [4] or toric hyperkähler, by R. Bielawski and A. Dancer [6]. The odd-dimensional counterpart, namely the compact contact toric manifolds, are classified by E. Lerman [20], whereas toric Sasaki manifolds were also studied by M. Abreu [3], [7]. These were used to produce examples of compact Sasaki-Einstein manifolds, for instance by D. Martelli, J. Sparks and S.-T. Yau [22], A. Futaki, H. Ono and G. Wang [12], C. van Coevering [28].

In the present paper, we consider toric geometry in the context of locally conformally Kähler manifolds. These are defined as complex manifolds admitting a compatible metric, which, on given charts, is conformal to a local Kähler metric. The differentials of the logarithms of the conformal factors glue up to a well-defined closed 1-form, called the Lee form. We are mostly interested in the special class of so-called Vaisman manifolds, defined by the additional property of having parallel Lee form. By analogy to the other geometries, we introduce the notion of toric locally conformally Kähler manifold. More precisely, we require the existence of an effective torus action of dimension half the dimension of the manifold, which preserves the holomorphic structure and is twisted Hamiltonian. I. Vaisman [27] introduced twisted Hamiltonian actions and they have been used for instance by S. Haller and T. Rybicki [18] and by R. Gini, L. Ornea and M. Parton [15], where reduction results for locally symplectic, respectively locally conformal Kähler manifolds are given, or more recently by A. Otiman [24].

Vaisman geometry is closely related to both Sasaki and Kähler geometry. In fact, a locally conformally Kähler manifold may be equivalently defined as a manifold whose universal covering is Kähler and on which the fundamental group acts by holomorphic homotheties. For Vaisman manifolds, the universal and the minimal covering are Kähler cones over Sasaki manifolds, as proven in [25], [16]. On the other hand, in the special case of strongly regular compact Vaisman manifolds, the quotient by the 2-dimensional distribution spanned by the Lee and anti-Lee vector fields is a Kähler manifold, *cf.* [26].

The purpose of this paper is to make a first step towards a possible classification of toric Vaisman, or more generally, toric locally conformally Kähler manifolds, by showing that the above mentioned connections between Vaisman and Sasaki, respectively Kähler manifolds are still true when imposing the toric condition. For the precise statements of these equivalences, we refer to Theorem 5.26, Theorem 5.28 and Theorem 5.29.

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2 Preliminaries

A *locally conformally Kähler manifold* (shortly lcK) is a conformal Hermitian manifold $(M^{2n}, [g], J)$ of complex dimension $n \geq 2$, such that for one (and hence for all) metric g in the conformal class, the corresponding fundamental 2-form $\omega := g(\cdot, J\cdot)$ satisfies: $\mathrm{d}\omega = \theta \wedge \omega$, with θ a closed 1-form, called the *Lee form* of the Hermitian structure (g, J). Equivalently, there exists an atlas on M, such that the restriction of g to any chart is conformal to a Kähler metric. In fact, the differential of the logarithmic of the conformal factors are, up to a constant, equal to the Lee form. It turns out to be convenient to denote also by (M, g, J, θ) an lcK manifold, when fixing one metric g in the conformal class. By ∇ we denote the Levi-Civita connection of g.

We denote by $\theta^\sharp$ the vector field dual to θ with respect to the metric g, the so-called *Lee vector field* of the lcK structure, and by $J\theta^\sharp$ the *anti-Lee vector field.*

Remark 5.1 *On an lcK manifold (M, g, J, θ), the following formula for the covariant derivative of J holds:*

$$2\nabla_X J = X \wedge J\theta^\sharp + JX \wedge \theta^\sharp, \quad \forall X \in \mathfrak{X}(M),$$

or, more explicitly, applied to any vector field $Y \in \mathfrak{X}(M)$:

$$2(\nabla_X J)(Y) = \theta(JY)X - \theta(Y)JX + g(JX, Y)\theta^\sharp + g(X, Y)J\theta^\sharp.$$

In particular, it follows that $\nabla_{\theta^\sharp} J = 0$ and $\nabla_{J\theta^\sharp} J = 0$.

Remark 5.2 *On an lcK manifold (M^{2n}, g, J, θ), a vector field X preserving the fundamental 2-form ω, also preserves the Lee form,* i.e. *$\mathcal{L}_X \omega = 0$ implies $\mathcal{L}_X \theta = 0$, as follows. As the differential and the Lie derivative with respect to a vector field commute to each other,* e.g. *by the Cartan formula, we obtain:*

$$0 = d(\mathcal{L}_X \omega) = \mathcal{L}_X(d\omega) = \mathcal{L}_X(\theta \wedge \omega) = \mathcal{L}_X \theta \wedge \omega + \theta \wedge \mathcal{L}_X \omega = \mathcal{L}_X \theta \wedge \omega.$$

Since the map form $\Omega^1(M)$ to $\Omega^3(M)$ given by wedging with ω is injective, for complex dimension $n \geq 2$, it follows that $\mathcal{L}_X \theta = 0$.

We now recall the definition of Vaisman manifolds, which were first introduced and studied by I. Vaisman [25], [26]:

Definition 5.3 *A Vaisman manifold is an lcK manifold (M, g, J, θ) admitting a metric in the conformal class, such that its Lee form is parallel,* i.e. *$\nabla\theta = 0$.*

Note that on a compact lcK manifold, a metric with parallel Lee form θ, if it exists, is unique up to homothety in its conformal class and coincides with the so-called *Gauduchon metric*, *i. e.* the metric with co-closed Lee form: $\delta\theta = 0$. In this paper, we scale any Vaisman metric g such that the norm of its Lee vector field $\theta^\sharp$, which is constant since θ is parallel, equals 1.

Definition 5.4 *The automorphism group of a Vaisman manifold (M, g, J, θ) is denoted by a slight abuse of notation* $\mathrm{Aut}(M) := \mathrm{Aut}(M, g, J, \theta)$ *and is defined as the group of conformal biholomorphisms:*

$$\mathrm{Aut}(M) = \{F \in \mathrm{Diff}(M) \mid F^*J = J, [F^*g] = [g]\}.$$

We emphasize here that we define the group of automorphisms like for lcK manifolds, namely we do not ask for the automorphisms of a Vaisman manifold to be isometries of the Vaisman metric, but only to preserve its conformal class. Hence, the Lie algebra of $\mathrm{Aut}(M)$ is:

$$\mathfrak{aut}(M) = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X J = 0, \mathcal{L}_X g = fg, \text{ for some } f \in \mathcal{C}^\infty(M)\}. \tag{5.1}$$

We denote by $\mathfrak{isom}(M)$ and $\mathfrak{hol}(M)$ the Lie algebras of Killing vector fields with respect to the Vaisman metric g, respectively of holomorphic vector fields.

The following lemma collects some known properties of Vaisman manifolds that are used in the sequel.

Lemma 5.5 () *On a Vaisman manifold (M, J, g, θ), the Lee vector field $\theta^\sharp$ and the anti-Lee vector field $J\theta^\sharp$ are both holomorphic Killing vector fields:*

$$\mathcal{L}_{\theta^\sharp} g = 0, \quad \mathcal{L}_{\theta^\sharp} J = 0, \quad \mathcal{L}_{J\theta^\sharp} g = 0, \quad \mathcal{L}_{J\theta^\sharp} J = 0,$$

so that $\theta^\sharp, J\theta^\sharp \in \mathfrak{isom}(M) \cap \mathfrak{hol}(M)$. In particular, it follows that $\mathcal{L}_{\theta^\sharp}\omega = 0$, $\mathcal{L}_{J\theta^\sharp}\omega = 0$ and $\theta^\sharp$ commutes with $J\theta^\sharp$:

$$[\theta^\sharp, J\theta^\sharp] = J[\theta^\sharp, \theta^\sharp] = 0.$$

The Lee vector field $\theta^\sharp$ and the anti-Lee vector field $J\theta^\sharp$ span a 1-dimensional complex Lie subalgebra of $\mathfrak{isom}(M) \cap \mathfrak{hol}(M)$, which moreover lies in the center of $\mathfrak{isom}(M) \cap \mathfrak{hol}(M)$. Thus, these vector fields give rise to the so-called canonical foliation $\mathcal{F}$ by 1-dimensional complex tori on a Vaisman manifold, which is a totally geodesic Riemannian foliation.

Proof.

The following relation holds for any integrable complex structure J: $\mathcal{L}_{JX}J = J \circ \mathcal{L}_X J$, for all vector fields X. This shows that a vector field X is holomorphic if and only if JX is holomorphic. In particular, we obtain that $J\theta^\sharp$ is a holomorphic vector field, because $\theta^\sharp$ is holomorphic. We now show that $J\theta^\sharp$ is a Killing vector field. By the formula in Remark 5.1, we obtain for $Y = \theta^\sharp$ and $X \in \mathfrak{X}(M)$:

$$\begin{aligned} 2(\nabla_X J)(\theta^\sharp) &= -\theta(\theta^\sharp)JX + \theta(J\theta^\sharp)X + g(JX, \theta^\sharp)\theta^\sharp + g(X, \theta^\sharp)J\theta^\sharp \\ &= -JX - g(X, J\theta^\sharp)\theta^\sharp + g(X, \theta^\sharp)J\theta^\sharp. \end{aligned} \tag{5.2}$$

Hence, it follows that $\nabla J\theta^\sharp = (\nabla J)(\theta^\sharp)$ (since $\theta^\sharp$ is parallel) is skew-symmetric:

$$\begin{aligned} 2g((\nabla_X J)(\theta^\sharp), Y) &= -g(JX, Y) - g(X, J\theta^\sharp)g(Y, \theta^\sharp) + g(X, \theta^\sharp)g(Y, J\theta^\sharp) \\ &= g(JY, X) + g(Y, J\theta^\sharp)g(X, \theta^\sharp) - g(Y, \theta^\sharp)g(X, J\theta^\sharp) \\ &= -2g((\nabla_Y J)(\theta^\sharp), X). \end{aligned} \tag{5.3}$$

Thus, $J\theta^\sharp$ is a Killing vector field of the Vaisman metric g. In particular, it follows:

$$\{\theta^\sharp, J\theta^\sharp\} \subset \mathfrak{isom}(M) \cap \mathfrak{hol}(M).$$

Let X be a holomorphic Killing vector field. Thus, X also preserves the fundamental form ω and by Remark 5.2 it holds: $\mathcal{L}_X\theta = 0$. Furthermore, the following general equality $(\mathcal{L}_X g)(\cdot, \theta^\sharp) = \mathcal{L}_X\theta - g(\cdot, \mathcal{L}_X\theta^\sharp)$ shows that

$$[X, \theta^\sharp] = \mathcal{L}_X\theta^\sharp = 0.$$

Since X is holomorphic, it also follows that

$$[X, J\theta^\sharp] = J[X, \theta^\sharp] = 0.$$

Thus, $\{\theta^\sharp, J\theta^\sharp\}$ is a subset of the center of $\mathfrak{isom}(M) \cap \mathfrak{hol}(M)$.

Recall that the *cone* over a Riemannian manifold (W, g_W) is defined as $\mathcal{C}(W) := \mathbb{R} \times W$ with the metric $g_{cone} := 4e^{-2t}(dt^2 + p^*g_W)$, where t is the parameter on $\mathbb{R}$ and

$p\colon \mathbb{R}\times W \to W$ is the projection on W. We denote the radial flow on the cone by φ, *i.e.* $\varphi_s\colon \mathcal{C}(W)\to\mathcal{C}(W)$, $\varphi_s(t,w) := (t+s,w)$, for each $s\in\mathbb{R}$. Note that this definition is equivalent, up to a constant factor[1], to the more common definition in the literature, namely $(\mathbb{R}_+\times W, dr^2 + r^2 p^* g_W)$ via the change of variable $r := e^{-t}$.

We have the following result:

Proposition 5.6 *For any complete Riemannian manifold (W, g_W), each homothety of the Riemannian cone $(\mathcal{C}(W) = \mathbb{R}\times W, 4e^{-2t}(dt^2 + p^* g_W))$ is of the form $(t,w)\mapsto (t+\rho, \psi(w))$, where $e^{-2\rho}$ is the dilatation factor and ψ is an isometry of (W, g_W). In particular, all the isometries of the cone $\mathcal{C}(W)$ come from isometries of W.*

Proposition 5.6 is proved in [16] for the compact case, *i.e.* when (W, g_W) is a compact Riemannian manifold. In [5], F. Belgun and A. Moroianu extended this result to the larger class of so-called cone-like manifolds. For the complete case, Proposition 5.6 can be proven as follows, after making, for convenience, the coordinate change $r = 2e^{-t}$. It is known that the metric completion of the cone $(\mathbb{R}_+\times W, dr^2 + r^2 p^* g_W)$ is a metric space obtained by adding a single point. It can further be showed that the incomplete geodesics of the cone are exactly its rays, $r\mapsto (r,w)$, for any fixed $w\in W$. Since the image of an incomplete geodesic through any isometry is again an incomplete geodesic, it follows that each isometry φ of the cone preserves the radial vector field, *i.e.* $\varphi_*\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r}$. For any vector field X tangent to W, $\varphi_*(X)$ is also tangent to W, since it is orthogonal to $\varphi_*\left(\frac{\partial}{\partial r}\right)$ and $\varphi_*\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r}$. Altogether, this implies that φ has the following form: $(r,w)\mapsto (r,\psi(w))$, with ψ an isometry of W. The statement for the homotheties then also follows, by composing for instance any isometry with the following homothety of the cone: $(r,w)\mapsto (e^{-\rho} r, w)$, for some $\rho\in\mathbb{R}$.

Definition 5.7 *A Sasaki structure on a Riemannian manifold (W, g_W) is a complex structure J on its cone $\mathcal{C}(W)$, such that (g_{cone}, J) is Kähler and for all $\lambda\in\mathbb{R}$, the homotheties $\rho_\lambda\colon \mathcal{C}(W)\to\mathcal{C}(W)$, $\rho_\lambda(t,w) := (t+\lambda, w)$ are holomorphic.*

There are many equivalent definitions of a Sasaki structure, see *e.g.* the monography [7]. In particular, a Sasaki manifold W^{2n+1} is endowed with a metric g_W and a contact 1-form η (*i.e.* $\eta\wedge(\mathrm{d}\eta)^n \neq 0$), whose Reeb vector field ξ, which is defined by the equations $\eta(\xi) = 1$ and $\iota_\xi \mathrm{d}\eta = 0$, is Killing. The restriction of the endomorphism $\Phi := \nabla^{g_W}\xi$ to the contact distribution $\ker(\eta)$ defines an integrable CR-structure. When needed, we may write a Sasaki structure by denoting all these data, as $(W, g_W, \xi, \eta, \Phi)$. We recall also that an automorphism of a Sasaki manifold is an isometry which preserves the Reeb vector field (see *e.g.* [7, Lemma 8.1.15]).

Definition 5.8 *A toric Sasaki manifold is a Sasaki manifold (W^{2m+1}, g_W, ξ) endowed with an effective action of the torus T^{m+1}, which preserves the Sasaki structure and such that ξ is an element of the Lie algebra $\mathfrak{t}_{m+1}$ of the torus. Equivalently, a toric Sasaki manifold is a Sasaki manifold whose Kähler cone is a toric Kähler manifold.*

[1]We choose to multiply the metric by the constant factor 4, so that later on a Vaisman manifold obtained as a quotient of a cone, the Vaisman metric has the property that its Lee vector field is of length 1 (see Section 5).

Thus, toric Sasaki manifolds are in particular contact toric manifolds of Reeb type. E. Lerman classified compact connected contact toric manifolds, see [20, Theorem 2.18].

One way to construct Vaisman manifolds, *cf.* R. Gini, L. Ornea, M. Parton, [15, Proposition 7.3], is the following:

Proposition 5.9 *Let W be a Sasaki manifold and Γ be a group of biholomorphic homotheties of the Kähler cone $\mathcal{C}(W)$, acting freely and properly discontinuously on $\mathcal{C}(W)$, such that Γ commutes with the radial flow generated by $\frac{\partial}{\partial t}$. Then $M := \mathcal{C}(W)/\Gamma$ has a naturally induced Vaisman structure.*

Conversely, it is implicitly proved in the work [25] of I. Vaisman that the universal covering of a Vaisman manifold, endowed with the Kähler metric, is a cone over a Sasaki manifold. R. Gini, L. Ornea, M. Parton and P. Piccinni showed in [16] that this result is true for any presentation of a Vaisman manifold. Let us recall that a *presentation* of a locally conformally Kähler manifold is a pair (K, Γ), where K is a homothetic Kähler manifold, *i.e.* the Kähler metric is defined up to homotheties, and Γ is a discrete group of biholomorphic homotheties acting freely and properly discontinuously on K. To any presentation, there is a group homomorphism, which associates to each homothety, its dilatation factor, $\rho_K \colon \Gamma \to \mathbb{R}^+$, such that $\gamma^*(g_K) = \rho_K(\gamma) g_K$, for any $\gamma \in \Gamma$, where g_K is the Kähler metric (up to homothety) on K. The *maximal presentation* is the universal covering $(\widetilde{M}, \pi_1(M))$ of the lcK manifold M and the *minimal presentation* or *minimal covering* is given by

$$(\widehat{M} := \widetilde{M}/(\mathrm{Isom}(\widetilde{M}, g_K) \cap \pi_1(M)), \Gamma_{\min} := \pi_1(M)/(\mathrm{Isom}(\widetilde{M}, g_K) \cap \pi_1(M)).$$

Hence, each $\gamma \in \Gamma_{\min} \setminus \{\mathrm{id}\}$ acts as a proper homothety on $(\widehat{M}, g_K)$, *i.e.* γ is not an isometry: $\rho_{\widehat{M}}(\gamma) \neq 1$. We can now state the following consequence of [16, Theorem 4.2]:

Proposition 5.10 *The minimal (resp. universal) covering of a Vaisman manifold (M, g, J, θ) is biholomorphic and conformal to the Kähler cone of a Sasaki manifold and $\Gamma_{\min}$ (resp. $\pi_1(M)$) acts on the cone by biholomorphic homotheties with respect to the Kähler cone metric.*

Remark 5.11 *On the minimal covering $\widehat{M}$ of an lcK manifold (M, g, J, θ), the pull-back $\hat{\theta}$ of the Lee form is exact. This property is clearly true on the universal covering $\widetilde{M}$, since the pull-back of θ is still closed, hence exact: $\tilde{\theta} = \mathrm{d}f$, as $\widetilde{M}$ is simply-connected. The minimal covering is obtained from the universal covering by quotiening out the isometries (of the Kähler metric $e^{-f}\tilde{g}$ on $\widetilde{M}$) in $\pi_1(M)$. Therefore, the function f projects onto a function $\hat{f} \in \mathcal{C}^\infty(\widehat{M})$, such that $\hat{\theta} = \mathrm{d}\hat{f}$.*

3 Twisted Hamiltonian Actions on lcK manifolds

Let (M, g, J, θ) be an lcK manifold. We consider d^θ, the so-called *twisted differential*, defined by:

$$\mathrm{d}^\theta \colon \Omega^*(M) \longrightarrow \Omega^{*+1}(M), \quad \mathrm{d}^\theta \alpha := d\alpha - \theta \wedge \alpha.$$

Remark that $\mathrm{d}^\theta \circ \mathrm{d}^\theta = 0$ if and only if $\mathrm{d}\theta = 0$ and that in this case d^θ anti-commutes with d. By definition, on an lcK manifold, we have $\mathrm{d}^\theta \omega = 0$.

Let $\mathcal{L}^\theta$ denote the *twisted Lie derivative* defined by $\mathcal{L}^\theta_X := \mathrm{d}^\theta \circ \iota_X + \iota_X \circ \mathrm{d}^\theta$. The following relation holds, for any $X \in \mathfrak{X}(M)$:

$$\mathcal{L}^\theta_X \omega = \mathcal{L}_X \omega - \theta(X)\omega, \tag{5.4}$$

as follows from the following direct computation:

$$\begin{aligned}\mathcal{L}_X \omega &= d\iota_X \omega + \iota_X d\omega = d\iota_X\omega + \iota_X(\theta \wedge \omega) \\ &= d\iota_X\omega + \theta(X)\omega - \theta \wedge \iota_X \omega = \mathrm{d}^\theta(\iota_X \omega) + \theta(X)\omega \\ &= \mathcal{L}^\theta_X \omega + \theta(X)\omega.\end{aligned}$$

Definition 5.12 *Given a function $f \in C^\infty(M)$ on an lcK manifold (M, g, J, θ), its* associated twisted Hamiltonian vector field X_f *is defined as the ω-dual of* $\mathrm{d}^\theta f = df - f\theta$, i. e. $\iota_{X_f}\omega = \mathrm{d}^\theta f$. *The subset* $\mathrm{Ham}^\theta(M) \subset \mathfrak{X}(M)$ *of* twisted Hamiltonian vector fields *is that of vector fields on M admitting such a presentation.*

We remark that the notion of twisted Hamiltonian vector field is invariant under conformal changes of the metric, even though the function associated to a twisted Hamiltonian vector field changes by the conformal factor. More precisely, if $g' = e^\alpha g$, then $\omega' = e^\alpha \omega$, $\theta' = \theta + d\alpha$ and the following relation holds $d^{\theta'} f = e^\alpha \mathrm{d}^\theta(e^{-\alpha} f)$. Note that if M is not globally conformally Kähler, the map $C^\infty(M) \to \mathfrak{X}(M)$, $f \mapsto X_f$ is injective.

Lemma 5.13 *Twisted Hamiltonian vector fields on an lcK manifold (M, g, J, θ) have the following properties:*

(i) $\forall X \in \mathrm{Ham}^\theta(M)$ *the following relations hold:*

$$\mathcal{L}^\theta_X \omega = \mathrm{d}^\theta \iota_X \omega = 0.$$

(ii) $\mathrm{Ham}^\theta(M)$ *is a vector subspace of* $\mathfrak{X}(M)$, *such that* $J\theta^\sharp \in \mathrm{Ham}^\theta(M)$ *and* $\theta^\sharp \notin \mathrm{Ham}^\theta(M)$.

(iii) Twisted Hamiltonian vector fields do not leave invariant the fundamental form ω, but conformally invariant, i.e. for all $X \in \mathrm{Ham}^\theta(M)$:

$$\mathcal{L}_X \omega = \theta(X)\omega. \tag{5.5}$$

Proof. (i) follows directly by the Cartan formula.
(ii) This can be seen as follows:

$$(\iota_{J\theta^\sharp}\omega)(X) = \omega(J\theta^\sharp, X) = g(J\theta^\sharp, JX) = \theta(X),$$

hence $\iota_{J\theta^\sharp}\omega = \theta = \mathrm{d}^\theta f$, where f is the constant function equal to -1.

However, the Lee vector field $\theta^\sharp$ is not twisted Hamiltonian, since the 1-form $\iota_{\theta^\sharp}\omega$ is not even d^θ-closed:

$$\mathrm{d}^\theta \iota_{\theta^\sharp}\omega = \mathcal{L}^\theta_{\theta^\sharp}\omega - \iota_{\theta^\sharp}\mathrm{d}^\theta\omega = \mathcal{L}_{\theta^\sharp}\omega - \theta(\theta^\sharp)\omega = -\omega \neq 0$$

(iii) follows from (5.4).

The *twisted Poisson bracket* on $C^\infty(M)$ is defined by:

$$\{f_1, f_2\} := \omega((\mathrm{d}^\theta f_1)^\sharp, (\mathrm{d}^\theta f_2)^\sharp) = \omega(X_{f_1}, X_{f_2}). \tag{5.6}$$

and it turns $C^\infty(M)$ into a Lie algebra. Hence, the following equality holds: $X_f = -J((\mathrm{d}^\theta f)^\sharp)$.

Recall that for any action of a Lie group G on a manifold M, and for any X in the Lie algebra of G, it is naturally associated the so-called *fundamental vector field* X_M, defined as $X_M(x) = \frac{d}{dt}|_{t=0}(\exp(tX)\cdot x)$, for any $x \in M$, where $\exp\colon \mathfrak{g} \to G$ is the exponential map of the group G and " $\cdot$ " denotes the action of G on M. We identify the elements of the Lie algebra with the induced fundamental vector fields, when there is no ambiguity.

Definition 5.14 *Let (M, g, J, θ) be a locally conformally Kähler manifold with fundamental 2-form ω. The action of a Lie group G on M is called*

- weakly twisted Hamiltonian *if the associated fundamental vector fields are twisted Hamiltonian,* i. e. *there exists a linear map*

  $$\mu : \mathfrak{g} \to C^\infty(M)$$

  such that $\iota_X\omega = \mathrm{d}^\theta\mu^X$, for all fundamental vector fields $X \in \mathfrak{g}$. This means that the twisted Hamiltonian vector field associated to $\mu^X := \mu(X)$ is exactly the vector field X. The condition here is that all fundamental vector fields are twisted Hamiltonian: $\mathfrak{g} \subseteq \mathrm{Ham}^\theta(M)$.

- twisted Hamiltonian *if the map μ can be chosen to be a Lie algebra homomorphism with respect to the Poisson bracket defined in* (5.6).
  In this case the Lie algebra homomorphism μ is called a momentum map *for the action of G.*

The map μ may equivalently be considered as the map:

$$\mu\colon M \to \mathfrak{g}^*, \quad \langle\, \mu(x),\, X\,\rangle := \mu^X(x),\ \forall X \in \mathfrak{g}, \forall x \in M.$$

The condition on $\mu\colon \mathfrak{g} \to C^\infty(M)$ to be a homomorphism of Lie algebras is equivalent to $\mu\colon M \to \mathfrak{g}^*$ being equivariant with respect to the adjoint action on $\mathfrak{g}^*$, the dual of the Lie algebra of G.

Remark 5.15 *The property of an action to be twisted Hamiltonian is a property of the conformal structure, even though the Poisson structure on $\mathcal{C}^\infty(M)$ is not conformally invariant. If $g' = e^\alpha g$, then $\mu^{\theta'} = e^\alpha\mu^\theta$.*

We define toric lcK manifolds by analogy to other toric geometries, as follows:

Definition 5.16 *A connected locally conformally Kähler manifold* $(M, [g], J)$ *of dimension* $2n$ *equipped with an effective holomorphic and twisted Hamiltonian action of the standard (real)* n*-dimensional torus* T^n*:*

$$\tau : T^n \to \mathrm{Diff}(M)$$

is called a toric locally conformally Kähler manifold*.*

Remark 5.17 *Let* (M, g, J, θ) *be a Vaisman manifold. Let* $\pi : \widetilde{M} \to M$ *be the universal covering of* M*. We still denote by* J *the induced complex structure on* $\widetilde{M}$*. Since* $\pi^*\theta$ *is closed and* $\widetilde{M}$ *simply-connected, it follows that it is also exact,* i.e. *there exists* $h \in \mathcal{C}^\infty(\widetilde{M})$ *such that* $\pi^*\theta = dh$*. We define the following metric and associated fundamental form on* $\widetilde{M}$*, which build a Kähler structure:*

$$\tilde{g} := e^{-h}\pi^* g, \quad \widetilde{\omega} = e^{-h}\pi^*\omega.$$

This can be showed as follows:

$$d\widetilde{\omega} = d(e^{-h}\pi^*\omega) = e^{-h}(-dh \wedge \pi^*\omega + \pi^* d\omega) = e^{-h}(-dh \wedge \pi^*\omega + \pi^*\theta \wedge \pi^*\omega) = 0.$$

A twisted Hamiltonian G*-action on* (M, g, J, ω) *is equivalent to a Hamiltonian* $\widetilde{G}^0$*-action on the Kähler manifold* $(\widetilde{M}, \tilde{g}, J, \widetilde{\omega})$*, where* $\widetilde{G}^0$ *denotes the identity component of the universal covering of* G*. To check this it is sufficient to consider the infinitesimal action of the Lie algebra* $\mathfrak{g}$*. We identify* $X \in \mathfrak{g} = \mathrm{Lie}(G)$ *with its associated fundamental vector field on* M*. Then, the fundamental vector field associated to* $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$ *equals* $\pi^* X$*. If* $\mu : \mathfrak{g} \to \mathcal{C}^\infty(M)$ *is the momentum map of the twisted Hamiltonian action of* G *on* M*, then the map*

$$\widetilde{\mu} : \mathfrak{g} \to \mathcal{C}^\infty(\widetilde{M}), \quad X \mapsto e^{-h}\pi^*\mu^X$$

is the momentum map of the Hamiltonian action of $\widetilde{G}$ *on* $\widetilde{M}$ *with respect to the Kähler form* $\widetilde{\omega}$*. In fact, for any* $X \in \mathfrak{g}$*, we have* $\iota_X\omega = \mathrm{d}^\theta\mu^X$*, by definition of a twisted Hamiltonian action. We now compute on* $\widetilde{M}$*:*

$$\begin{aligned}\iota_{\pi^*X}\widetilde{\omega} &= \iota_{\pi^*X}(e^{-h}\pi^*\omega) = e^{-h}\pi^*(\iota_X\omega) = e^{-h}\pi^*(\mathrm{d}^\theta\mu^X)\\ &= e^{-h}\pi^*(d\mu^X - \mu^X\theta) = e^{-h}(d(\pi^*\mu^X) - \pi^*\mu^X \cdot dh) = d(e^{-h}\pi^*\mu^X).\end{aligned}$$

Since μ *is a homomorphism of Poisson algebras, the same holds for* $\widetilde{\mu}$*.*

4 Toric Vaisman manifolds

In this section (M, g, J, θ) denotes a Vaisman manifold.

Remark 5.18 *Let* X *be a holomorphic Killing vector field on a Vaisman manifold* (M, J, g, θ)*. The* 1*-form* $\iota_X\omega$ *is* d^θ*-closed iff* $\mathcal{L}^\theta_X\omega = 0$*. Since by assumption* $\mathcal{L}_X\omega = 0$*, it follows from* (5.4) *that* $\mathrm{d}^\theta\iota_X\omega = 0$ *iff* $\theta(X) = 0$*.*

Lemma 5.19 *Any twisted Hamiltonian holomorphic action of a Lie group G on a complete Vaisman manifold (M, g, J, θ) is automatically isometric with respect to the Vaisman metric and the following inclusion holds:*

$$\mathfrak{g} \subseteq \mathfrak{isom}(M) \cap \ker(\theta).$$

Proof. By assumption we have:

$$\mathfrak{g} \subseteq \mathfrak{hol}(M) \cap \mathrm{Ham}^{\theta}(M).$$

Let $X \in \mathfrak{g} \subseteq \mathrm{Ham}^{\theta}(M)$. From (5.5), it follows that $\mathcal{L}_X \omega = \theta(X)\omega$. Since X is a holomorphic vector field, we have $\mathcal{L}_X J = 0$. From the following relation between the Lie derivatives:

$$(\mathcal{L}_X \omega)(\cdot, \cdot) = (\mathcal{L}_X g)(\cdot, J\cdot) + g(\cdot, (\mathcal{L}_X J)\cdot), \quad \forall X \in TM,$$

it follows that $\mathcal{L}_X g = \theta(X)g$. Hence, X is a conformal vector field.

In order to show that X is a Killing vector field, we consider the universal covering $\widetilde{M}$ of M, which by Proposition 5.10, is biholomorphic conformal to the Kähler cone $(\mathcal{C}(\widetilde{W}), 4e^{-2t}(\mathrm{d}t^2 + p^* g_{\widetilde{W}}))$ over a Sasaki manifold $(\widetilde{W}, g_{\widetilde{W}})$. The lift of X to $\widetilde{M}$, denoted by $\widetilde{X}$, is a conformal Killing vector field with respect to the pull-back metric $\tilde{g} := \pi^* g$, where π denotes the natural projection from $\widetilde{M}$ to M. Moreover, we claim that $\widetilde{X}$ is a Killing vector field with respect to the Kähler cone metric g_K, as the following computation shows:

$$\begin{aligned}\mathcal{L}_{\widetilde{X}} g_K &= \mathcal{L}_{\widetilde{X}}(e^{-2t}\tilde{g}) = \widetilde{X}(e^{-2t})\tilde{g} + \mathcal{L}_{\widetilde{X}}(e^{-2t}\tilde{g}) = e^{-2t}(-2\mathrm{d}t(\widetilde{X})\tilde{g} + \pi^*(\mathcal{L}_X g)) \\ &= e^{-2t}(-2\mathrm{d}t(\widetilde{X}) + \pi^*(\theta(X)g)) = 0,\end{aligned}$$

where we used that $\pi^*\theta = 2\mathrm{d}t$.

Let us note, that by the theorem of Hopf-Rinow, the assumption on (M, g) to be complete implies that $(\widetilde{M}, \tilde{g})$ is complete and further that also $(\widetilde{W}, g_{\widetilde{W}})$ is complete. Thus, applying Proposition 5.6, it follows that all Killing vector fields of the cone metric on $\mathcal{C}(\widetilde{W})$ are projectable onto Killing vector fields of the Sasaki metric $g_{\widetilde{W}}$ on $\widetilde{W}$. In particular, it follows that $\left[\widetilde{X}, \frac{\partial}{\partial t}\right] = 0$. This implies that $[X, \theta^\sharp] = 0$.

From $\mathcal{L}_X g = \theta(X)g$ and $[X, \theta^\sharp] = 0$, it follows:

$$\theta(X)\theta = \mathcal{L}_X \theta = \mathrm{d}(\theta(X)),$$

where the last equality is obtained by Cartan formula. This identity applied to $\theta^\sharp$ yields $\theta^\sharp(\theta(X)) = \theta(X)$. On the other hand, we compute:

$$\theta^\sharp(\theta(X)) = (\mathcal{L}_{\theta^\sharp}\theta)(X) + \theta([\theta^\sharp, X]) = 0,$$

since again by the Cartan formula, we have $\mathcal{L}_{\theta^\sharp}\theta = \mathrm{d}\iota_{\theta^\sharp}\theta = 0$. Therefore, we obtain $\theta(X) = 0$. Hence, X is a Killing vector field of g and is orthogonal to $\theta^\sharp$.

Remark 5.20 *On a compact Vaisman manifold, A. Moroianu and L. Ornea [23] proved that every conformal Killing vector field is a Killing vector field with respect to the Vaisman metric. Recently, P. Gauduchon and A. Moroianu [14] proved the following more general result: on a connected compact oriented Riemannian manifold admitting a non-trivial parallel vector field, any conformal Killing vector field is Killing.*

The next result shows that an action preserving the whole Vaisman structure is automatically twisted hamiltonian. Moreover, it shows that the momentum map is given by the anti-Lee 1-form. More precisely, we have:

Lemma 5.21 *Let (M, g, J, θ) be a Vaisman manifold and X be a holomorphic Killing vector field on M, which is in the kernel of θ. Then X is a twisted Hamiltonian vector field with Hamiltonian function $f := J\theta(X)$,* i.e. *the following equality holds:*

$$\iota_X \omega = \mathrm{d}f - f\theta = \mathrm{d}^\theta f.$$

Proof. We compute the differential of f as follows:

$$\mathrm{d}f(Y) = (\nabla_Y J)(\theta \wedge X) + g(J\theta, \nabla_Y X) = \frac{1}{2}(f\theta(Y) + \omega(X, Y)) - g(Y, \nabla_{J\theta} X) \qquad (5.7)$$

since $\theta(X) = 0$, $|\theta| = 1$ and ∇X is skew-symmetric. Note that

$$-\nabla_{J\theta} X = \mathcal{L}_X J\theta - \nabla_X J\theta = -\nabla_X J\theta = \frac{1}{2}(f\theta + JX),$$

since X preserves θ and J, and θ is parallel. Substituting in (5.7), we obtain

$$\mathrm{d}f = \frac{1}{2}(f\theta + \iota_X\omega) - \nabla_{J\theta} X = f\theta + \iota_X\omega.$$

We consider now toric Vaisman manifolds, as a special class of toric lcK manifolds, *cf.* Definition 5.16.

Remark 5.22 *As in the case of Kähler toric manifolds, where a Hamiltonian holomorphic action automatically preserves the Kähler metric, also on Vaisman manifolds a twisted Hamiltonian holomorphic action preserves the Vaisman metric, see Lemma 5.19.*

Remark 5.23 *The maximal dimension of an effective twisted Hamiltonian torus action on a $2n$-dimensional Vaisman manifold is n. The proof follows by the same argument as for Hamiltonian actions on symplectic manifolds (for details see* e.g. *[9]).*

In particular, on a toric Vaisman manifold, by Lemma 5.19, the following inclusions hold:

$$\mathfrak{t}_n \subseteq \mathfrak{hol}(M) \cap \mathrm{Ham}^\theta(M) \subseteq \mathfrak{isom}(M) \cap (\theta^\sharp)^\perp,$$

where $\mathfrak{t}_n$ denotes the Lie algebra of T^n. Moreover, we show the following:

Lemma 5.24 *On a toric Vaisman manifold (M^{2n}, g, J, θ), the anti-Lee vector field is part of the twisted Hamiltonian torus action,* i.e. $J\theta^\sharp \in \mathfrak{t}_n$.

Proof. We argue by contradiction and assume that $J\theta^\sharp \notin \mathfrak{t}_n$. By Remark 5.23, n is the maximal dimension of a torus acting effectively and twisted Hamiltonian on a $2n$-dimensional Vaisman manifold. Hence, it suffices to show that $\mathfrak{t}_n \cup \{J\theta^\sharp\}$ is still an abelian Lie algebra acting twisted Hamiltonian on (M, g, J). This is a consequence of Lemma 5.13, (ii), stating that $J\theta^\sharp \in \mathrm{Ham}^\theta M$, and of Lemma 5.5, where it is shown that $J\theta^\sharp$ is in the center of $\mathfrak{aut}(M)$, so in particular commutes with all elements of $\mathfrak{t}_n \subset \mathfrak{aut}(M)$.

Example 5.25 *The standard example of a Vaisman manifold is $S^1 \times S^{2n-1}$, endowed with the complex structure and metric induced by the diffeomorphism*

$$\Psi\colon \mathbb{C}^n \setminus \{0\}/\mathbb{Z} \longrightarrow S^1 \times S^{2n-1}, \ \ [z] \longmapsto (e^{i \ln |z|}, \frac{z}{|z|}),$$

where $[z] = [z']$ if and only if there exists $k \in \mathbb{Z}$ such that $z' = e^{2\pi k} z$. The Hermitian metric $\frac{\mathrm{d}z_j \otimes \mathrm{d}\bar{z}_j}{|z|^2}$ on $\mathbb{C}^n \setminus \{0\}$ is invariant under the action of $\mathbb{Z}$ and hence it descends to a Hermitian metric on $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}$, with

$$g := \mathrm{Re}\frac{\mathrm{d}z_j \otimes \mathrm{d}\bar{z}_j}{|z|^2}, \quad \omega := i\frac{\mathrm{d}\bar{z}_j \wedge \mathrm{d}z_j}{|z|^2}, \quad \theta := \mathrm{d}\ln |z|^{-2}$$

defining the lcK metric, 2-fundamental form and Lee form respectively. The Lee form θ is parallel, hence $S^1 \times S^{2n-1}$ is Vaisman. We define the following T^n action on $\mathbb{C}^n \setminus \{0\}/\mathbb{Z} \simeq S^1 \times S^{2n-1}$: $t \longmapsto ([z] \mapsto [t_1 z_1, \dots, t_n z_n])$. It is easy to check that it is effective and holomorphic. A basis of fundamental vector fields of the action is given by $X_j([z]) := i z_j \frac{\partial}{\partial z_j} - i\bar{z}_j \frac{\partial}{\partial \bar{z}_j}$, for $j \in \{1, \dots, n\}$. Therefore, we have

$$\iota_{X_j}\omega = \frac{\bar{z}_j \mathrm{d}z_j + z_j \mathrm{d}\bar{z}_j}{|z|^2} = \mathrm{d}\left(\frac{|z_j|^2}{|z|^2}\right) - \frac{|z_j|^2}{|z|^2}\theta = \mathrm{d}^\theta\left(\frac{|z_j|^2}{|z|^2}\right),$$

which shows that the action is also twisted Hamiltonian, with momentum map $\mu^{X_j}([z]) := \frac{|z_j|^2}{|z|^2}$. Hence, $S^1 \times S^{2n-1}$ is a toric Vaisman manifold.

We are now ready to state our result:

Theorem 5.26 *Let (M^{2n}, J, g, θ) be a complete Vaisman manifold and W^{2n-1} be its associated Sasaki manifold, such that the minimal covering $\widehat{M}$ of M is biholomorphic and conformal to the Kähler cone over W. If M is a toric Vaisman manifold, then W is a toric Sasaki manifold.*

Proof. Let $\pi\colon \widehat{M} \to M$ denote the projection of the minimal covering. $\widehat{M}$ is naturally endowed with the pullback metric $\hat{g} := \pi^* g$ and the complex structure $\widehat{J}$, such that π is a local isometric biholomorphism. By Proposition 5.10, we know that $\widehat{M} = \mathcal{C}(W)$, where $(W, g_W, \xi, \eta, \Phi)$ is a complete Sasaki manifold and $\mathcal{C}(W) = \mathbb{R} \times W$ is its cone. By this identification, the lift of the 1-form θ, which is exact on $\widehat{M}$ *cf.* Remark 5.11, equals $2\mathrm{d}t$, where t is the coordinate of the factor $\mathbb{R}$. Moreover, the relationship between the induced metric $\hat{g}$ and the Kähler cone metric $g_K := g_{\mathrm{cone}} = 4e^{-2t}(dt^2 + p^* g_W)$ is the following: $\hat{g} = e^{2t} g_K = 4(\mathrm{d}t^2 + p^* g_W)$. The complex structure $\widehat{J}$ coincides with the usual complex structure induced by the Sasaki structure on the Kähler cone, *i.e.* $\widehat{J}(p^*\xi) = \frac{\partial}{\partial t}$ and $\widehat{J}(\frac{\partial}{\partial t}) = -p^*\xi$ and $\widehat{J}$ coincides with the transverse complex structure Φ on the horizontal distribution $\xi^\perp$ in $\mathrm{T}W$.

By assumption, the Vaisman manifold M^{2n} is toric, hence it is equipped with an effective twisted Hamiltonian action $\tau : T^n \to \mathrm{Diff}(M)$.

By Lemma 5.24, $J\theta^\sharp \in \mathfrak{t}_n$, hence we may choose a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{t}_n \cong \mathbb{R}^n$, such that the fundamental vector field on M associated to X_1 is $-2J\theta^\sharp$. For simplicity, we still denote the induced fundamental vector fields of the action by $X_j \in \mathfrak{X}(M)$, for $1 \leq j \leq n$. From Remark 5.2 and Lemma 5.19, it follows that $[X_j, \theta^\sharp] = 0$, for all j. Moreover, Lemma 5.19 implies that $g(X_j, \theta^\sharp) = 0$, for all j.

Since π is a local diffeomorphism, we can lift each vector field X_j uniquely to a vector field $\widehat{X_j}$ on $\widehat{M}$. As X_j commute pairwise, the same is true for their lifts, *i.e.* we have $[\widehat{X_j}, \widehat{X_k}] = 0$, for all $1 \leq j, k \leq n$. Note that the lift of $\theta^\sharp$ to $\widehat{M}$ equals $(\pi^*\theta)^{\sharp,\hat{g}}$, *i.e.* the dual of $\pi^*\theta = 2\mathrm{d}t$ with respect to the metric $\tilde{g}$. Thus, $\widehat{\theta^\sharp} = \frac{1}{2}\frac{\partial}{\partial t}$. Hence, $\widehat{X_1}$, which is the lift to the universal covering of $-2J\theta^\sharp$, equals $\widehat{X_1} = -2\widehat{J}((\pi^*\theta)^{\sharp,\hat{g}}) = -2\widehat{J}(\frac{1}{2}\frac{\partial}{\partial t}) = p^*\xi$. It is then clear, that $\widehat{X_1}$ projects through p on W to ξ. In fact, each lift $\widehat{X_j}$ is projectable onto vector fields on W, since they are constant along $\mathbb{R}$: $\left[\frac{\partial}{\partial t}, \widehat{X_j}\right] = 2\left[\widehat{\theta^\sharp}, \widehat{X_j}\right] = 2\widehat{[\theta^\sharp, X_j]} = 0$, for all $1 \leq j \leq n$. We denote their projections by $Y_j := p_*\widehat{X_j} \in \mathfrak{X}(W)$, for all $1 \leq j \leq n$. In particular, Y_1 equals the Reeb vector field ξ. We remark that the vector fields $\{Y_1, \ldots, Y_n\}$ commute pairwise and are complete. Thus, they give rise to an effective action $\tau_W \colon \mathbb{R}^n \to \mathrm{Diff}(W)$.

We show next that this action acts by automorphisms of the Sasaki structure, *i.e.* the vector fields $\{Y_1, \ldots, Y_n\}$ preserve (g_W, ξ, η, Φ). More precisely, it is sufficient to show that they are Killing vector fields, since they commute with $Y_1 = \xi$. The torus action τ on the Vaisman manifold is by isometries, *cf.* Lemma 5.19, so each vector field X_j is Killing with respect to the Vaisman metric g. The projection π being a local isometry between $(\widehat{M}, \hat{g})$ and (M, g), it follows that each lift $\widehat{X_j}$ is a Killing vector field with respect to the metric $\hat{g} = 4(\mathrm{d}t^2 + p^* g_W)$, hence their projections onto (W, g_W) are still Killing vector fields with respect to the Sasaki metric g_W.

In order to conclude that W is a toric Sasaki manifold, we need to argue why the action τ_W of $\mathbb{R}^n$ on W naturally induces an action of the torus T^n on W. For this, it is enough to show that for each $X \in \mathfrak{t}_n$ with the associated fundamental vector on M having closed orbits of the same period, also its lift $\widehat{X}$ has the same property. Let Ψ_s and $\widehat{\Psi}_s$ denote the flow of X, resp. of $\widehat{X}$. We assume that all the orbits of X close after time $s = 1$, *i.e.* $\Psi_1(x) = x$, for all $x \in M$. Then, since $\pi \circ \widehat{\Psi}_s = \Psi_s \circ \pi$, for all values of s, it follows that for each $(t, w) \in \widehat{M}$, we have $\pi((t, w)) = \pi(\Psi_1(t, w))$, so there exists $\gamma \in \Gamma_{\min}$, such that $\Psi_1(t, w) = \gamma \cdot (t, w)$. A priori γ may depend on the choice of (t, w), but since the function defined in this way would be continuous with values in the discrete group $\Gamma_{\min}$, it must be constant. On the other hand, $\widehat{X}$ is a Killing vector field with respect to the Kähler metric $g_K = e^{-2t}\hat{g}$ (since, as shown above, $\widehat{X}$ is Killing with respect to the pull-back $\hat{g}$ of the Vaisman metric and it also leaves invariant the conformal factor, as $\theta(X) = 0$ implies $\mathrm{d}t(\widehat{X}) = 0$). Applying Proposition 5.6, we have $\Psi_1(t, w) = (t, \psi(w))$, where ψ is an isometry of W. Hence, for all $(t, w) \in \widehat{M}$, we have: $\gamma \cdot (t, w) = (t, \psi(w))$, in particular γ acts as an isometry of the Kähler metric. As $\gamma \in \Gamma_{\min}$, it follows from the definition of $\Gamma_{\min}$ that $\gamma = \mathrm{id}$. We conclude that all orbits of $\widehat{X}$ are also closed with the same period.

Remark 5.27 *The argument in the proof of Theorem 5.26 also applies to the universal covering of a toric complete Vaisman manifold, showing that it carries an effective holomorphic Hamiltonian action of $\mathbb{R}^n$,* i.e. *a completely integrable Hamiltonian system, with respect to its Kähler structure.*

Conversely, we show:

Theorem 5.28 *Let $(W^{2n-1}, g_W, \xi, \eta, \Phi)$ be a toric complete Sasaki manifold with the effective torus action $\tau \colon T^n \to \mathrm{Diff}(W)$. Let Γ be a discrete group of biholomophic homotheties acting freely and properly discontinuously on the Kähler cone $(\mathcal{C}(W), 4e^{-2t}(dt^2 + p^*g_W))$ and hence inducing a Vaisman structure on the quotient $M := \mathcal{C}(W)/\Gamma$. If the action τ, which is extended to $\mathcal{C}(W)$ acting trivially on $\mathbb{R}$, commutes with Γ, then M is a toric Vaisman manifold.*

Proof.

Let π denote the projection corresponding to the action of the group Γ, $\pi \colon \mathcal{C}(W) \to M = \mathcal{C}(W)/\Gamma$. We still denote by τ the extension of the T^n-action on $\mathbb{R} \times W$, obtained by letting T^n act trivially on $\mathbb{R}$. The assumption on this action to commute with the group Γ, ensures that it naturally projects onto a T^n-action on the quotient M, that we denote by $\bar{\tau} \colon T^n \to \mathrm{Diff}(M)$. In order to show that M is a toric Vaisman manifold, we need to check that this action is effective, holomorphic and twisted Hamiltonian with respect to the induced Vaisman structure on M^{2n}, that we denote by (g, J, θ). For this, we study the induced fundamental vector fields of the action.

We recall that through π, the exact form $2\mathrm{d}t$ projects to the Lee form θ of the induced Vaisman structure on the quotient M, the vector field $\frac{1}{2}\frac{\partial}{\partial t}$ projects to the Lee vector field $\theta^\sharp$ and hence, $p^*\xi = -J(\frac{\partial}{\partial t})$ projects to $-2J\theta^\sharp$. We also know that the metric which projects onto the Vaisman metric g is the product metric $4(\mathrm{d}t^2 + g_W)$.

By assumption, the action $\tau \colon T^n \to \mathrm{Diff}(W)$ preserves the Sasaki structure and has the property that $\xi \in \mathfrak{t}_n$. We choose a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{t}_n \cong \mathbb{R}^n$, such that the fundamental vector field on M associated to X_1 is $-\frac{1}{2}\xi$. We still denote the induced fundamental vector fields of the action by $X_j \in \mathfrak{X}(W)$, for $1 \le j \le n$. We consider the pull-back of these vector fields on $\mathcal{C}(W) = \mathbb{R} \times W$ through the projection $p \colon \mathbb{R} \times W \to W$, $Y_j := p^*X_j$, for $1 \le j \le n$. Hence, $Y_1 = -p^*(\frac{1}{2}\xi)$ projects through π to $J\theta^\sharp$, which is a Killing vector field with respect to the Vaisman metric g. We claim that this property is true for all vector fields Y_j. First we notice that each Y_j

is projectable through π, because of the hypothesis on the action of T^n on $\mathbb{R} \times W$ to commute with Γ. We denote the projected vector fields by $\overline{Y_j} := \pi_* Y_j$. In fact, by construction, the vector fields $\{\overline{Y_1}, \dots, \overline{Y_n}\}$ are exactly the fundamental vector fields of the action $\bar{\tau}$ corresponding to the fixed basis $\{X_1, \dots, X_n\}$ of $\mathfrak{t}_n$. As each X_j is a Killing vector field with respect to the Sasaki metric g_W, it follows that $Y_j = p^* X_j$ is a Killing vector field with respect to the product metric $4(\mathrm{d}t^2 + g_W)$ on $\mathbb{R} \times W$. Since π is a local isometry between $(\mathbb{R} \times W, 4(\mathrm{d}t^2 + g_W))$ and (M, g), the vector fields $\overline{Y_j}$ are still Killing with respect to g.

The same argument works in order to show that $\overline{Y_j}$ is a holomorphic vector field on the Vaisman manifold (M, g, J, θ). Namely, we use that π is a local biholomorphism and that $Y_j = p^* X_j$ is a holomorphic vector field with respect to the induced holomorphic structure J_K on the cone $\mathcal{C}(W)$. The last statement follows from the following straightforward computation:

$$
\begin{aligned}
(\mathcal{L}_{Y_j} J_K)(p^*\xi) &= \left[Y_j, \frac{\partial}{\partial t}\right] - J_K(p^*([X_j, \xi])) = 0, \\
(\mathcal{L}_{Y_j} J_K)\left(\frac{\partial}{\partial t}\right) &= -[Y_j, p^*\xi] - J_K\left(\left[Y_j, p^* \frac{\partial}{\partial t}\right]\right) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{L}_{Y_j} J_K)(p^* X) &= [Y_j, p^*(\Phi(X))] - J_K([Y_j, p^* X]) \\
&= p^*([X_j, X] - \Phi[X_j, X]) = p^*((\mathcal{L}_{X_j}\Phi)(X)) = 0,
\end{aligned}
$$

where X is any vector field of the contact distribution of W, *i.e.* X is orthogonal to ξ.

The Kähler form of the Kähler cone is on the one hand given by: $\omega_K = e^{-2t}\pi^*\omega$, where ω is the fundamental 2-form of the Vaisman structure: $\omega = g(\cdot, J\cdot)$, and on the other hand, it is exact and equals $2\mathrm{d}(e^{-2t}p^*\eta)$, where η is the 1-form of the Sasaki structure on W. We recall that any action preserving the Sasaki structure is automatically hamiltonian with momentum map obtained by pairing with the opposite of the 1-form η, *i.e.* for any Killing vector field X commuting with ξ, we have: $\iota_X d\eta = -d(\eta(X))$. The induced action on the Kähler cone is also hamiltonian. More precisely, the following equation holds, for X as above:

$$
\iota_{(p^*X)}\omega_K = 2\iota_{(p^*X)}\mathrm{d}(e^{-2t}p^*\eta) = -2\mathrm{d}(e^{-2t}\eta(X) \circ p),
$$

since $\mathcal{L}_X(e^{-2t}\eta) = 0$. We compute for each $1 \le j \le n$:

$$
\begin{aligned}
\pi^*(\iota_{\overline{Y_j}}\omega) = \iota_{Y_j}(\pi^*\omega) = \iota_{Y_j}(e^{2t}\omega_K) &= e^{2t}\mathrm{d}(e^{-2t}\eta(X_j) \circ p) \\
&= -2(\eta(X_j) \circ p)\mathrm{d}t + d(\eta(X_j) \circ p).
\end{aligned}
\tag{5.8}
$$

We now notice that the function $\eta(X_j) \circ p$ is invariant under the action of the group Γ. This is due, on the one hand, to the fact that X_j is a fundamental vector field associated to the torus action, which by assumption commutes with Γ. On the other hand, η equals the projection on W of $J_K(\mathrm{d}t)$ and Γ acts by biholomorphisms with respect to J_K and by homotheties with respect to the cone metric, hence it automatically commutes with the radial flow φ_s, *cf.* Proposition 5.6. We denote the projection of the function $\pi_*(\eta(X_j)\circ p)$ on M by $\mu(X_j) = -\frac{1}{2}J\theta(\overline{Y_j})$. Hence, the right hand side of (5.8) is projectable through π and we obtain on M:

$$\iota_{\overline{Y_j}}\omega = -\mu(X_j)\theta + d(\mu(X_j)) = \mathrm{d}^\theta(\mu(X_j)),$$

showing that the torus action $\bar{\tau}$ defined above is twisted Hamiltonian with momentum map defined by $\mu\colon \mathfrak{t}_n \to \mathcal{C}^\infty(M)$, $\mu(X) = -\frac{1}{2}J\theta(X_M)$, where X_M is the fundamental vector field associated to X through the action $\bar{\tau}$.

5 Toric Compact Regular Vaisman manifolds

We consider in this section the special case of a strongly compact regular Vaisman manifold and show that it is toric if and only if the Kähler quotient is toric.

Recall that a Vaisman manifold (M, g, J, θ) is called *regular* if the 1-dimensional distribution spanned by the Lee vector field $\theta^\sharp$ is regular, meaning that $\theta^\sharp$ gives rise to an S^1-action on M, and it is called *strongly regular* if both the Lee and anti-Lee vector field give rise to an S^1-action.

Theorem 5.29 *Let (M^{2n}, g, J) be a strongly regular compact Vaisman manifold and denote by $\overline{M} := M/\{\theta^\sharp, J\theta^\sharp\}$ the quotient manifold with the induced structures $\bar{g}$ and $\bar{J}$. Then M is a toric Vaisman manifold if and only if $\overline{M}$ is a toric Kähler manifold.*

Proof. Let (M, g, J) be a strictly regular compact Vaisman manifold with Lee vector field $\theta^\sharp$ and let $\pi\colon M \to \overline{M}$ denote the projection onto the quotient manifold. By definition, each of the vector fields $\theta^\sharp$ and $J\theta^\sharp$ gives rise to a circle action, which by Lemma 5.5 are both holomorphic and isometric. In this case, the metric g and the complex structure J descends through π to the quotient manifold $\overline{M}$ and it is a well-known result that $(\overline{M}, \bar{g}, \bar{J})$ is a compact Hodge manifold and π is a Riemannian submersion. Moreover, the curvature form of the connection 1-form $\theta - iJ\theta$ of the principal bundle $\pi\colon M \to \overline{M}$ projects onto the Kähler form $\overline{\omega}$ of $\overline{M}$. For details on these results see *e.g.* [26] or [11, Theorem 6.3].

Let us first assume that (M^{2n}, g, J) is a compact toric Vaisman manifold, *i.e.* there is an effective twisted Hamiltonian holomorphic action $\tau : T^n \to \mathrm{Diff}(M)$ with momentum map $\mu\colon M \to \mathbb{R}^n$. We show that there is a naturally induced effective Hamiltonian action on the compact Kähler quotient, $\bar{\tau}\colon T^{n-1} \to \mathrm{Diff}(\overline{M})$, where T^{n-1} is the torus obtained from T^n by quotiening out the direction corresponding to the circle action of $J\theta^\sharp$, which by Lemma 5.24 lies in the torus. Hence, by the definition of T^{n-1}, we have the following relation between the Lie algebras of the corresponding tori: $\mathfrak{t}_n = \mathfrak{t}_{n-1} \oplus \langle J\theta^\sharp \rangle$. Since $\{\theta^\sharp, J\theta^\sharp\}$ is included in the center of $\mathfrak{aut}(M)$, *cf.* Lemma 5.5, it follows that the action of T^n on M naturally descends through π to an action on the quotient $\overline{M} = M/\{\theta^\sharp, J\theta^\sharp\}$. By Remark 5.22, the action of T^n on (M, g, J) is both holomorphic and isometric. Thus, the action induced by T^n on $(\overline{M}, \bar{g}, \bar{J})$ is also isometric and holomorphic. This can be checked as follows. By the definition of the induced action, the fundamental vector fields X_M and $X_{\overline{M}}$, associated to the action of any $X \in \mathfrak{t}_{\mathfrak{n}}$ on M, respectively on $\overline{M}$, are related by $\pi_*(X_M) = X_{\overline{M}}$. Since the Lie derivative commutes with the pull-back, we have $\mathcal{L}_{X_{\overline{M}}}\bar{g} = \mathcal{L}_{\pi_*(X_M)}\pi_*g = \pi_*(\mathcal{L}_{X_M}g) = 0$ and similarly we obtain $\mathcal{L}_{X_{\overline{M}}}\bar{J} = 0$. This action of T^n on $\overline{M}$ is not effective, since it is trivial in the direction of the anti-Lee vector field $J\theta^\sharp \in \mathfrak{t}_n$. However, as $\theta^\sharp \notin \mathfrak{t}_n$ by Lemma 5.19, it follows that the restriction of this action to the above defined "complementary" torus T^{n-1} of $J\theta^\sharp$ in T^n is effective. We denote it by $\bar{\tau}\colon T^{n-1} \to \mathrm{Diff}(\overline{M})$ and we only need to check that it is a Hamiltonian action on the symplectic manifold $(\overline{M}, \overline{\omega})$. For this, we consider the map $p \circ \mu\colon M \to \mathbb{R}^{n-1}$, where $p\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection from $\mathfrak{t}_n$ to $\mathfrak{t}_{n-1}$, corresponding to quotiening out $\langle J\theta^\sharp \rangle$, through the identification of the deals of the Lie algebras of T^n and T^{n-1} with $\mathbb{R}^n$, respectively $\mathbb{R}^{n-1}$. It follows that for any $X \in \mathfrak{t}_{n-1}$, the function $(p \circ \mu)(X)\colon M \to \mathbb{R}$ is invariant under the action of $\theta^\sharp$ and of $J\theta^\sharp$, hence it is projectable through π onto a function on $\overline{M}$, which we denote $\bar{\mu}^X$. It turns out that $\bar{\mu}$ is the momentum map of the action $\bar{\tau}$. In order to show this, let $\bar{x} \in \overline{M}$, $\overline{Y} \in T_{\bar{x}}\overline{M}$, choose $x \in M$ with $\pi(x) = \bar{x}$ and let $Y := (d_x\pi|_{\langle \theta^\sharp, J\theta^\sharp \rangle^\perp})^{-1}(\overline{Y})$. We compute at the point $\bar{x}$, for all $X \in \mathfrak{t}_{n-1}$: $(\iota_{X_{\overline{M}}}\overline{\omega})(\overline{Y}) = (\pi_*\omega)(\pi_*X_M, \pi_*Y)$ and on the other hand:

$$d_{\bar{x}}\bar{\mu}^X(\overline{Y}) = d_x\mu^X(Y) = \mathrm{d}^\theta_x\mu^X(Y) = (\iota_{X_M}\omega)_x(Y) = \omega_x(X_{Mx}, Y),$$

thus proving that $\iota_{X_{\overline{M}}}\overline{\omega} = d\bar{\mu}^X$, for all $X \in \mathfrak{t}_{n-1}$. Note that by definition $\bar{\mu}$ inherits from μ the property of being a Lie algebra homomorphism from $\mathbb{R}^{n-1}$ endowed with the trivial Lie bracket to $\mathcal{C}^\infty(\overline{M})$ endowed with the Poisson bracket. Concluding, we have shown that the induced action $\bar{\tau}\colon T^{n-1} \to \mathrm{Diff}(\overline{M})$ is an effective holomorphic Hamiltonian action on $\overline{M}$, so $(\overline{M}, \bar{g}, \bar{J})$ is a compact Kähler toric manifold.

Conversely, let us assume that (M^{2n}, g, J) is a strictly regular compact Vaisman manifold, such that the Kähler quotient $(\overline{M}, \bar{g}, \bar{J})$ is a toric Kähler manifold. We show that (M, g, J) is then a toric Vaisman manifold. By assumption, $\overline{M}^{2n-2}$ is equipped with an effective Hamiltonian holomorphic action of T^{n-1}, which we denote by $\bar{\tau}\colon T^{n-1} \to \mathrm{Diff}(\overline{M})$. We also denote by $\bar{\mu}\colon \overline{M} \to \mathbb{R}^{n-1}$ one of the momentum maps of this action, which is unique up to an additive constant. Let $X_1, \ldots, X_{n-1}$ be a basis of the Lie algebra $\mathfrak{t}_{n-1} \cong \mathbb{R}^{n-1}$ and $f_j := \mu^{X_j}$ the corresponding Hamiltonian functions on $\overline{M}$, *i.e.* $\iota_{\overline{X}_j}\overline{\omega} = df_j$, for $j \in \{1, \ldots, n\}$, where for simplicity we denote the fundamental vector fields of the action on $\overline{M}$ by $\overline{X}_j = X_j^{\overline{M}} \in \mathfrak{X}(M)$. We now consider the horizontal distribution $\mathcal{D}$ defined for each $x \in M$ by $\mathcal{D}_x := \{X \in T_xM \mid X \perp \theta^\sharp_x, X \perp J\theta^\sharp_x\}$ and the corresponding horizontal pull-back Y_j of each vector field X_j through π, *i.e.* for each $x \in M$, $(Y_j)_x := ((d_x\pi)|_{\mathcal{D}_x})^{-1}((\overline{X_j})_{\pi(x)})$, for $j \in \{1, \ldots, n\}$. Each vector field Y_j is complete and thus gives rise to an action of $\mathbb{R}$ on M. In the sequel, we modify them in the direction of $J\theta^\sharp$ in order to obtain the lifted torus action on M.

Set $Z_j := Y_j - (f_j \circ \pi)J\theta^\sharp$, for $1 \leq j \leq n-1$.

Step 1. We first show that each of these vector fields commutes with $\theta^\sharp$ and with $J\theta^\sharp$ and preserves the distribution $\mathcal{D}$, since it leaves invariant both 1-forms θ and $J\theta$. Namely, we compute for $1 \leq j \leq n-1$:

$$[Z_j, \theta^\sharp] = \Big[Y_j - f_j \circ \pi \cdot J\theta^\sharp, \theta^\sharp\Big] = 0, \quad [Z_j, J\theta^\sharp] = \Big[Y_j - f_j \circ \pi \cdot J\theta^\sharp, J\theta^\sharp\Big] = 0, \qquad (5.9)$$

where we use that all functions $f_j \circ \pi$ and all horizontal lifts Y_j are by definition constant along the flows of $\theta^\sharp$ and of $J\theta^\sharp$. We further obtain by applying the Cartan formula:

$$\mathcal{L}_{Z_j}\theta = \iota_{Z_j} d\theta + d(\iota_{Z_j}\theta) = 0,$$

since θ is closed and Z_j is by definition in the kernel of θ. Since the curvature of the connection 1-form of the principal bundle $\pi\colon M \to \overline{M}$ projects onto the Kähler form $\overline{\omega}$, we obtain:

$$\mathcal{L}_{Z_j}(J\theta) = \iota_{Z_j} d(J\theta) + d(\iota_{Z_j}(J\theta)) = \iota_{Z_j}\pi^*\overline{\omega} - d(f_j \circ \pi) = \pi^*(\iota_{\overline{X}_j}\overline{\omega}) - \pi^*(df_j) = 0.$$

Indeed, in order to obtain this last equality it was necessary to perturb the horizontal lifts Y_j in the direction of $J\theta^\sharp$. We claim that the set $\{Z_1, \cdots, Z_{n-1}\}$ gives rise to an effective action of $\mathbb{R}^{n-1}$ on M. For this, it is sufficient to check that all their Lie brackets vanish, so they are commuting vector fields. The property of being an effective action is a consequence of the effectiveness of the action of T^{n-1} on $\overline{M}$. For each $j, k \in \{1, \dots, n-1\}$ we compute:

$$\begin{aligned}
[Z_j, Z_k] &= \Big[Y_j - f_j \circ \pi \cdot J\theta^\sharp, Y_k - f_k \circ \pi \cdot J\theta^\sharp\Big] \\
&= [Y_j, Y_k] - Y_j\,(f_k \circ \pi)\, J\theta^\sharp + Y_k\,(f_j \circ \pi)\, J\theta^\sharp \\
&= [\overline{X}_j, \overline{X}_k]^* - \big(\overline{\omega}(\overline{X}_j, \overline{X}_k) - df_j(\overline{X}_k) + df_k(\overline{X}_j)\big) \circ \pi \cdot J\theta^\sharp \\
&= [\overline{X}_j, \overline{X}_k]^* - \overline{\omega}(\overline{X}_k, \overline{X}_j) \circ \pi \cdot J\theta^\sharp = 0,
\end{aligned} \tag{5.10}$$

where by * is denoted the horizontal lift to the distribution $\mathcal{D}$ of a vector field on $\overline{M}$. For the above equalities we used again that $J\theta^\sharp(f_j \circ \pi) = 0$ and $[Y_j, J\theta^\sharp] = 0$, for all $1 \leq j \leq n-1$, as well as the fact that the curvature of the connection 1-form of the principal bundle equals $\pi^*\overline{\omega}$. Note that the last equality in (5.10) follows from the commutation of the vector fields $\overline{X}_j$ induced by the torus action and from the property of the momentum map $\bar{\mu}\colon \mathbb{R}^{n-1} \to \mathcal{C}^\infty(\overline{M})$ of being a Lie algebra homomorphism, where $\mathcal{C}^\infty(\overline{M})$ is endowed with the Poisson bracket. Hence, for any j, k, the following relations hold: $0 = \bar{\mu}([X_j, X_k]) = \{\bar{\mu}^{X_j}, \bar{\mu}^{X_k}\} = \overline{\omega}(\overline{X}_j, \overline{X}_k)$.

Step 2. We now check that the above defined action of $\mathbb{R}^{n-1}$ on M is holomorphic, isometric and twisted Hamiltonian. For each $1 \leq j \leq n-1$, Z_j is a holomorphic vector field by the following computation, which uses the commutation relations from (5.9):

$$(\mathcal{L}_{Z_j} J)(\theta^\sharp) = [Z_j, J\theta^\sharp] - J[Z_j, \theta^\sharp] = 0,\ (\mathcal{L}_{Z_j} J)(J\theta^\sharp) = -[Z_j, \theta^\sharp] - J[Z_j, J\theta^\sharp] = 0,$$

and since for each projectable horizontal vector field $Y \in \Gamma(\mathcal{D})$ with $\pi_* Y = \overline{Y}$ we have:

$$\begin{aligned}
(\mathcal{L}_{Z_j} J)(Y) =& [Y_j - (f_j \circ \pi) J\theta^\sharp, JY] - J[Y_j - (f_j \circ \pi) J\theta^\sharp, Y] \\
=& [Y_j, JY] - (f_j \circ \pi)[J\theta^\sharp, JY] + JY(f_j \circ \pi) J\theta^\sharp \\
& - J[Y_j, Y] + (f_j \circ \pi) J[J\theta^\sharp, Y] + Y(f_j \circ \pi)\theta^\sharp \\
=& [\overline{X}_j, \overline{J}\,\overline{Y}]^* - \overline{\omega}(\overline{X}_j, \overline{J}\,\overline{Y}) \circ \pi \cdot J\theta^\sharp + df_j(\overline{J}\,\overline{Y}) \circ \pi \cdot J\theta^\sharp \\
& - J([\overline{X}_j, \overline{Y}]^* - \overline{\omega}(\overline{X}_j, \overline{Y}) \circ \pi \cdot J\theta^\sharp) + df_j(\overline{Y}) \circ \pi \cdot \theta^\sharp \\
=& ([\overline{X}_j, \overline{J}\,\overline{Y}] - \overline{J}[\overline{X}_j, \overline{Y}])^* = 0,
\end{aligned}$$

where we use that by definition $\overline{J}$ is the projection of J, so $JY = (\overline{J}\,\overline{Y})^*$, as well as the equalities $[J\theta^\sharp, JY] = J[J\theta^\sharp, Y]$, following from the fact that $J\theta^\sharp$ is a holomorphic vector field, *cf.* Lemma 5.5. A similar computation yields that $\mathcal{L}_{Z_j} g = 0$, for $1 \leq j \leq n-1$, showing that each Z_j is a Killing vector field on M with respect to g. We further compute for each $j \in \{1, \dots, n-1\}$:

$$\begin{aligned}\iota_{Z_j}\omega &= \iota_{Y_j}\omega - (f_j \circ \pi)\iota_{J\theta^\sharp}\omega = \pi^*(\iota_{\overline{X}_j}\overline{\omega}) - (f_j \circ \pi)\theta \\ &= d(f_j \circ \pi) - (f_j \circ \pi)\theta = \mathrm{d}^\theta(f_j \circ \pi),\end{aligned}$$

by taking into account that $\omega = \pi^*\overline{\omega} - \theta \wedge J\theta$. This proves that each vector field Z_j is twisted Hamiltonian with respect to ω with Hamiltonian function $f_j \circ \pi$. Altogether, we have shown that the action of $\mathbb{R}^{n-1}$ defined by the vector fields $Z_1, \dots, Z_j$ respects the whole Vaisman structure (g, J, ω) and is twisted Hamiltonian with momentum map $\mu\colon \mathbb{R}^{n-1} \to \mathcal{C}^\infty(M)$, defined by $\mu^{X_j} := \bar{\mu}^{X_j} \circ \pi$, for $1 \leq j \leq n-1$. We also remark that μ is a Lie algebra homomorphism for $\mathcal{C}^\infty(M)$ endowed with the Poisson bracket defined by (5.6), as $\bar{\mu}$ is a Lie algebra homomorphism and the Poisson brackets are compatible through π. More precisely, we have for all $j, k \in \{1, \dots, n-1\}$:

$$\begin{aligned}\{\mu^{X_j}, \mu^{X_k}\} &\overset{(5.6)}{=} \omega(Z_j, Z_k) = \omega(Y_j - (f_j \circ \pi)J\theta^\sharp, Y_k - (f_k \circ \pi)J\theta^\sharp) = \\ &= \omega(Y_j, Y_k) - (f_j \circ \pi)\omega(J\theta^\sharp, Y_k) + (f_k \circ \pi)\omega(J\theta^\sharp, Y_j) \\ &= \overline{\omega}(\overline{X}_j, \overline{X}_k) \circ \pi - (f_j \circ \pi)\theta(Y_k) + (f_k \circ \pi)\,\theta(Y_j) \\ &= \{\bar{\mu}^{X_j}, \bar{\mu}^{X_k}\} \circ \pi = \bar{\mu}^{[X_j, X_k]} \circ \pi = \mu^{[X_j, X_k]}.\end{aligned}$$

Step 3. Next we show that the freedom in choosing the momentum map $\bar{\mu}$ of $\overline{M}$ up to an additive constant in $\mathbb{R}^{n-1}$ allows us to make the above defined action of $\mathbb{R}^{n-1}$ on M into an action of $T^{n-1} = S^1 \times \cdots \times S^1$ on M, thus lifting the action of T^{n-1} on $\overline{M}$. It is enough to establish this result for one direction and then apply it independently for each of the directions X_j, $j \in \{1, \dots, n-1\}$. The argument can be found in [13], Section 7.5., in the more general setting of lifting S^1-actions to Hermitian complex line bundles equipped with a $\mathbb{C}$-linear connection, whose curvature form is preserved by the circle action and in which case the property of being Hamiltonian is with respect to the closed curvature 2-form, which is not assumed to be symplectic. For the convenience of the reader we sketch here this argument using the above notation. Let X be a vector field on $\overline{M}$ which generates an S^1-action, whose orbits are assumed to have period 1. One considers the flow Φ_s of the horizontal lift of X to $M/\{\theta^\sharp\}$, which then satisfies: $\Phi_1(x) = \zeta(\bar{x}) \cdot x$, for all $x \in M/\{\theta^\sharp\}$, where $\bar{x}$ is the projection of x on $\overline{M}$ and $\zeta\colon \overline{M} \to S^1$. The main step is to show that ζ is constant. This follows from the following formula $\Phi_1 \cdot V = V - i\frac{\mathrm{d}\zeta(\overline{V})}{\zeta} J\theta^\sharp$, for all $\overline{V} \in \mathfrak{X}(\overline{M})$ and V its horizontal lift to $M/\{\theta^\sharp\}$, and from the fact that Φ_s preserves the horizontal distribution for all values of s. An S^1-action on $M/\{\theta^\sharp\}$ lifts to an S^1-action on M, since θ is closed, so its kernel defines an integrable distribution on M. Applied to our case, this result yields that for each $1 \leq j \leq n-1$, the Hamiltonian function $f_j \in \mathcal{C}^\infty(M)$ may be chosen (by adding an appropriate real constant, determined by ζ, which is well-defined up to an additive integer), such that the vector field $Z_j = Y_j - (f_j \circ \pi) \cdot J\theta^\sharp$ is the generator of an S^1-action on M, where Y_j denotes the horizontal lift of $\overline{X}_j$.

Since $J\theta^\sharp$ is a Killing, holomorphic, twisted Hamiltonian vector field lying in the center of $\mathfrak{aut}(M)$, *cf.* Lemma 5.5 and Lemma 5.13, it follows that the circle action induced by $J\theta^\sharp$ commutes with the circle actions induced by the above defined action of T^{n-1} on M, thus giving rise together to an effective holomorphic twisted Hamiltonian action of T^n on (M^{2n}, g, J), showing that this is a toric Vaisman manifold.

Applying this result to the Vaisman manifold $S^1 \times S^{2n-1}$ as described in Example 5.25, which is strongly regular, as $\theta^\sharp$ is the tangent vector to the S^1 factor and $J\theta^\sharp$ is tangent to the Hopf action on S^{2n-1}. This action commutes with the torus action defined in Example 5.25. It follows that this T^n-action descends to the quotient $(S^1 \times S^{2n-1})/\{\theta^\sharp, J\theta^\sharp\} \simeq \mathbb{C}P^n$.

Remark 5.30 *If (M^{2n}, g, J) is a regular compact Vaisman manifold, then the proof of Theorem 5.29 also shows that the following equivalence holds: M is a toric Vaisman manifold if and only if $M/\{\theta^\sharp\}$ is a toric Sasaki manifold. Again, this statement is only about the toric structure, since it is well-known that the quotient manifold is Sasaki (see* e.g. *[26] or [11]).*

We note that the proofs of the theorems of the last two sections are both constructive, showing that the torus actions on the Sasaki (respectively Kähler) manifold and on the Vaisman manifold naturally induce one another.

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Chapter 6

Remarks on the product of harmonic forms

Liviu Ornea and Mihaela Pilca

Abstract. A metric is formal if all products of harmonic forms are again harmonic. The existence of a formal metric implies Sullivan formality of the manifold, and hence formal metrics can exist only in presence of a very restricted topology. We show that a warped product metric is formal if and only if the warping function is constant and derive further topological obstructions to the existence of formal metrics. In particular, we determine necessary and sufficient conditions for a Vaisman metric to be formal.

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1 Introduction

A fundamental problem in algebraic topology is the reading of the homotopy type of a space in terms of cohomological data. A precise definition of this property was given by Sullivan in [S] and called *formality.* As concerns manifolds, it is known *e.g.* that all compact Riemannian symmetric spaces and all compact Kähler manifolds are formal. For a recent survey of topological formality, see [PS].

Sullivan also observed that if a compact manifold admits a metric such that the wedge product of any two harmonic forms is again harmonic, then, by Hodge theory, the manifold is formal. This motivated Kotschick to give the following:

Definition 6.1 *([K]) A Riemannian metric is called* (metrically) formal *if all wedge products of harmonic forms are harmonic.*

A closed manifold is called geometrically formal *if it admits a formal Riemannian metric.*

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In particular, the length of any harmonic form with respect to a formal metric is (pointwise) constant. This larger class of metrics having all harmonic (1-)forms of constant length naturally appears in other geometric contexts, for instance in the study of certain systolic inequalities, and has been investigated in [N], [NV].

Classical examples of geometrically formal manifolds are compact symmetric spaces. In [KT1] and [KT2] more general examples are provided, both of geometrically formal and of formal but non-geometrically formal homogeneous manifolds.

Geometric formality imposes strong restrictions on the (real) cohomology of the manifold. For example, it is proven in [K] that a manifold admits a non-formal metric if and only if it is not a rational homology sphere.

In this note, we shall obtain further obstructions to formality. We shall see (Section 2) that if a compact manifold with $b_1 = p \geq 1$ admits a formal metric, and if there exist two vanishing Betti numbers such that the distance between them is not larger than $p+2$, then all the intermediary Betti numbers must be zero too. Also, a conformal class of metrics on an even-dimensional compact manifold with non-zero middle Betti number can contain no more than one formal metric.

Our main concern will be the formality of warped products (Section 2). We shall show that a warped product metric on a compact manifold is formal if and only if the warping function is constant. On the way, we shall also provide a proof for the known fact (stated for instance in [K]) that a product of formal metrics is formal.

Unlike Kähler manifolds, which are known to be formal, for the time being, nothing is known about the Sullivan formality of locally conformally Kähler (in particular Vaisman) manifolds. In Section 3 of this note, we shall discuss compact Vaisman manifolds, whose universal cover is a special type of warped product, a Riemannian cone to be precise, and we shall find obstructions to the metric formality of a Vaisman metric. Several computational facts and their proofs are gathered in a final Appendix.

2 Geometric formality of warped product metrics

For the sake of completeness and as a first step in the study of geometrically formal warped products, we provide a proof for the formality of Riemannian product metrics.

Proposition 6.2 *If (M_1, g_1) and (M_2, g_2) are two compact Riemannian manifolds with formal metrics, then the metric $g = g_1 + g_2$ on the product manifold $M = M_1 \times M_2$ is also formal.*

Proof. Let $\gamma \in \Omega^p M$ and $\gamma' \in \Omega^q M$ be two harmonic forms on M. By Lemma 6.13, γ and γ' are given by linear combinations with real coefficients of the basis elements in (6.6). Thus, it is enough to check that the exterior product of any two such basis elements is a harmonic form on M. But:

$$(\pi_1^*(\alpha) \wedge \pi_2^*(\beta)) \wedge (\pi_1^*(\alpha') \wedge \pi_2^*(\beta')) = (-1)^{|\alpha'||\beta|} \pi_1^*(\alpha \wedge \alpha') \wedge \pi_2^*(\beta \wedge \beta'),$$

which is g-harmonic on M by Lemma 6.13 and by the formality of g_1 and g_2 (as $\alpha \wedge \alpha'$ is again a g_1-harmonic form and $\beta \wedge \beta'$ a g_2-harmonic form). $\square$

We now pass to the setting we are mainly interested in, namely the warped products.

Theorem 6.3 *Let (B^n, g_B) and (F^m, g_F) be two compact Riemannian manifolds with formal metrics. Then the warped product metric $g = \pi^*(g_B) + (\varphi \circ \pi)^2 \sigma^*(g_F)$ on $B \times_\varphi F$ is formal if and only if the warping function φ is constant.*

Proof. Let $\beta \in \Omega^p(F)$ be a g_F-harmonic form on F (as $b_m(F) = 1$, there exists at least a harmonic m-form on F). From (6.7) and (6.8), it follows that $\sigma^*\beta$ is a g-harmonic form on the warped product $B \times_\varphi F$. If we assume the warped metric g to be formal, it follows in particular that the length of $\sigma^*\beta$ is constant. As g_F is also assumed to be formal, the length of β is constant as well. On the other hand, the following relation holds:

$$g(\sigma^*\beta, \sigma^*\beta) = (\varphi \circ \pi)^{2p} g_F(\beta, \beta) \circ \sigma, \tag{6.1}$$

showing that the function φ must be constant.

Conversely, if φ is constant, then the warped product reduces to the Riemannian product between the Riemannian manifolds (B, g_B) and $(F, \varphi^2 g_F)$, which is geometrically formal by Proposition 6.2. □

Remark 6.4 *From the above proof we see that Theorem 6.3 holds more generally for metrics having all harmonic forms of constant length.*

An interesting question regarding the formal metrics is their existence in a given conformal class. Under a weak topological assumption, we prove that there may exist at most one such formal metric. More precisely, we have:

Proposition 6.5 *Let M^{2n} be an even-dimensional compact manifold whose middle Betti number $b_n(M)$ is non-zero. Then, in any conformal class of metrics there is at most one formal metric (up to homothety).*

Proof. Let $[g]$ be a class of conformal metrics on M and suppose there are two formal metrics g_1 and $g_2 = e^{2f} g_1$ in $[g]$. The main observation is that in the middle dimension the kernel of the codifferential is invariant at conformal changes of the metric, so that there are the same harmonic forms for all metrics in a conformal class: $\mathcal{H}^n(M, g_1) = \mathcal{H}^n(M, g_2)$. As $b_n(M) \geq 1$ there exists a non-trivial g_1-harmonic (and thus also g_2-harmonic) n-form α on M. The length of α must then be constant with respect to both metrics, which are assumed to be formal and thus we get:

$$g_2(\alpha, \alpha) = e^{2nf} g_1(\alpha, \alpha),$$

which shows that f must be constant.

□

Using the product construction to assure that the middle Betti number is non-zero, one can build such examples of formal metrics which are unique in their conformal class.

Other examples are provided by manifolds with "big" first Betti number, as follows from the following property of "propagation" of Betti numbers on geometrically formal manifolds proven in [K, Theorem 7]: *if* $b_1(M) = p \geq 1$, *then* $b_q(M) \geq \binom{p}{q}$, *for all* $1 \leq q \leq p$. In particular, if $b_1(M^{2n}) \geq n$, then $b_n(M^{2n}) \geq 1$.

Another property of the Betti numbers of geometrically formal manifolds is given by:

Proposition 6.6 *Let M^n be a compact geometrically formal manifold with $b_1(M) = p \geq 1$. If there exist two Betti numbers that vanish: $b_k(M) = b_{k+l}(M) = 0$, for some k and l with $0 < k + l < n$ and $0 < l \leq p + 1$, then all intermediary Betti numbers must vanish: $b_i(M) = 0$, for $k \leq i \leq k + l$. In particular, if there exists $k \geq \frac{n-p-1}{2}$ such that $b_k(M) = 0$, then $b_i(M) = 0$ for all $k \leq i \leq n - k$.*

Proof. Let $\{\theta_1, \ldots, \theta_p\}$ be an orthogonal basis of g-harmonic 1-forms, where g is a formal metric on M. We first notice that here is no ambiguity in considering the orthogonality with respect to the global scalar product or to the pointwise inner product, because, when restricting ourselves to the space of harmonic forms of a formal metric, these notions coincide. This is mainly due to [K, Lemma 4], which states that the inner product of any two harmonic forms is a constant function. Thus, if two harmonic forms α and β are orthogonal with respect to the global product, we get: $0 = (\alpha, \beta) = \int_M < \alpha, \beta > dvol_g =< \alpha, \beta > vol(M)$, showing that their pointwise inner product is the zero-function.

It is enough to show that $b_{k+1}(M) = 0$ and then use induction on i. Let α be a harmonic $(k+1)$-form. By formality, $\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_{l-1} \wedge \alpha$ is a harmonic $(k+l)$-form and thus must vanish, since $b_{k+l}(M) = 0$. On the other hand, $\theta_j^\sharp \lrcorner \alpha = (-1)^{k(n-k-1)} * (\theta_j \wedge * \alpha)$ is a harmonic k-form, again by formality. As $b_k(M) = 0$, it follows that $\theta_j^\sharp \lrcorner \alpha = 0$, for $1 \leq j \leq p$. Then, using that $\{\theta_1, \ldots, \theta_p\}$ are also orthogonal, we obtain:

$$0 = \theta_1^\sharp \lrcorner \cdots \lrcorner \theta_{l-1}^\sharp \lrcorner (\theta_1 \wedge \cdots \wedge \theta_{l-1} \wedge \alpha) = \pm |\theta_1|^2 \cdots |\theta_{l-1}|^2 \alpha,$$

which implies that $\alpha = 0$, because each θ_j has non-zero constant length. This shows that $b_{k+1}(M) = 0$. □

3 Geometric formality of Vaisman metrics

A Vaisman manifold is a particular type of locally conformal Kähler (LCK) manifold. It is defined as a Hermitian manifold (M, J, g), of real dimension $n = 2m \geq 4$, whose fundamental 2-form ω satisfies the conditions:

$$d\omega = \theta \wedge \omega, \quad \nabla \theta = 0.$$

Here θ is a (closed) 1-form, called the Lee form, and ∇ is the Levi-Civita connection of the LCK metric g (we always consider $\theta \neq 0$, to not include the Kähler manifolds among the Vaisman ones).

Locally, $\theta = df$ and the *local* metric $e^{-f} g$ is Kähler, hence the name LCK. When lifted to the universal cover, these local metrics glue to a global one, which is Kähler and acted on by homotheties by the deck group of the covering.

In the Vaisman case, the universal cover is a Riemannian cone. In fact, compact Vaisman manifolds are closely related to Sasakian ones, as the following structure theorem shows:

Theorem 6.7 *[OV1] Compact Vaisman manifolds are mapping tori over S^1. More precisely: the universal cover $\tilde{M}$ is a metric cone $N \times \mathbb{R}^{>0}$, with N compact Sasakian manifold and the deck group is isomorphic with $\mathbb{Z}$, generated by $(x, t) \mapsto (\lambda(x), t + q)$ for some $\lambda \in Aut(N)$, $q \in \mathbb{R}^{>0}$.*

This puts compact Vaisman manifolds into the framework of warped products and motivates their consideration here.

Vaisman manifolds are abundant. Every Hopf manifold (quotient of $\mathbb{C}^{\mathbb{N}} \setminus \{0\}$ by the cyclic group generated by a semi-simple operator with subunitary eigenvalues) is such, and all its compact complex submanifolds (see [Ve, Proposition 6.5]). Besides, the complete list of Vaisman compact surfaces is given in [B].

On the other hand, examples of LCK manifolds (satisfying only the condition $d\omega = \theta \wedge \omega$ for a closed θ) which cannot admit any Vaisman metric are also known, *e.g.* one type of Inoue surfaces and the non-diagonal Hopf surface, see [B]. The non-diagonal Hopf surface is particularly relevant for our discussion because it is topologically formal, as all manifolds having the same cohomology ring as a product of odd spheres.

For the time being, little information is known about the topology of LCK and, in particular, Vaisman manifolds. Below, we collect several known facts:

Theorem 6.8 *Let M be a compact Vaisman manifold.*
(i) Its fundamental group cannot be free non-abelian, [OV2].
(ii) $b_1(M)$ is odd, [V].

But nothing is known about the topological formality of LCK manifolds.

Being parallel and Killing (see [DO]), the Lee field $\theta^\sharp$ is real holomorphic and, together with $J\theta^\sharp$ generates a one-dimensional complex, totally geodesic, Riemannian foliation $\mathcal{F}$. Note that $\mathcal{F}$ is transversally Kähler, meaning that the transversal part of the Kähler form is closed (for a proof of this result, see *e*.g. [V, Theorem 3.1]).

In the sequel, the terms *basic (foliate)* and *horizontal* refer to $\mathcal{F}$. We recall that a form is called *horizontal* with respect to a foliation $\mathcal{F}$ if its interior product with any vector field tangent to the foliation vanishes and is called *basic* if in addition its Lie derivative along a vector field tangent to the foliation also vanishes. Moreover, we shall use the basic versions of the standard operators acting on $\Omega^*_B(M)$, the space of basic forms: Δ_B is the basic Laplace operator, L_B is the exterior multiplication with the transversal Kähler form and Λ_B its adjoint with respect to the transversal metric. For details on these operators and their properties we refer the reader to [T, Chapter 12].

The main result of this section puts severe restrictions on formal Vaisman metrics:

Theorem 6.9 *Let (M^{2m}, g, J) be a compact Vaisman manifold. The metric g is geometrically formal if and only if $b_p(M) = 0$ for $2 \leq p \leq 2m - 2$ and $b_1(M) = b_{2m-1}(M) = 1$,* i.e. *M is a cohomological Hopf manifold.*

Proof. Let $\gamma \in \Omega^p(M)$ be a harmonic form on M for some p, $1 \leq p \leq m-1$. By [V, Theorem 4.1], γ has the following form:

$$\gamma = \alpha + \theta \wedge \beta, \tag{6.2}$$

with α and β basic, transversally harmonic and transversally primitive.

Since α is basic, $J\alpha$ is also a basic p-form that is transversally harmonic and transversally primitive:

$$\Delta_B(J\alpha) = 0, \quad \Lambda_B(J\alpha) = 0,$$

because Δ_B and Λ_B both commute with the transversal complex structure J (as the foliation is transversally Kähler). Again from [V, Theorem 4.1], by taking $\beta = 0$, it follows that $J\alpha$ is a harmonic form on M: $\Delta(J\alpha) = 0$.

The assumption that g is geometrically formal implies that $\alpha \wedge J\alpha$ is harmonic on M, so that in particular it is coclosed: $\delta(\alpha \wedge J\alpha) = 0$. According to [V], this implies that $\alpha \wedge J\alpha$ is transversally primitive[2]: $\Lambda_B(\alpha \wedge J\alpha) = 0$.

On the other hand, it is proven in [GN, Proposition 2.2] that for primitive forms $\eta, \mu \in \Lambda^p V$, where (V, g, J) is any Hermitian vector space, the following algebraic relation holds:

$$(\Lambda)^p(\eta \wedge \mu) = (-1)^{\frac{p(p-1)}{2}} p! \langle \eta, J\mu \rangle, \tag{6.3}$$

where J is the extension of the complex structure to $\Lambda^* V$ defined by:

$$(J\eta)(v_1, \ldots, v_p) := \eta(Jv_1, \ldots, Jv_p), \quad \text{for all } \eta \in \Lambda^p V, v_1, \ldots, v_p \in V.$$

We apply the above formula to the transversal Kähler geometry and obtain that α vanishes everywhere:

$$0 = (\Lambda_B)^p(\alpha \wedge J\alpha) = (-1)^{\frac{p(p+1)}{2}} p! \langle \alpha, \alpha \rangle.$$

The same argument as above applied to $\beta \in \Omega_B^{p-1}(M)$ shows that β is identically zero if $p \geq 2$. Thus, $\gamma = 0$ for $2 \leq p \leq m-1$, which proves that:

$$b_2(M) = \cdots = b_{m-1}(M) = 0.$$

If $p = 1$, then β is a basic function, which is transversally harmonic, so that β is a constant. Thus γ is a multiple of θ, showing that the space of harmonic 1-forms on M is 1-dimensional: $b_1(M) = 1$.

It remains to show that the Betti number in the middle dimension, $b_m(M)$, also vanishes. This follows from Proposition 6.6 applied to $p = 1$, $k = m-1$ and $l = 2$.

The converse is clear, since the space of harmonic forms with respect to the Vaisman metric g is spanned by $\{1, \theta, *\theta, dvol_g\}$ and thus the only product of harmonic forms which is not trivial is $\theta \wedge *\theta = g(\theta, \theta) dvol_g$, which is harmonic because θ has constant length, being a parallel 1-form. $\square$

[2] We use a slightly different denomination as in [V], where instead the term *transversally effective* is used.

Remark 6.10 (i) *There exist Vaisman manifolds which do not admit any formal Vaisman metric. Indeed, let $f : N \hookrightarrow \mathbb{C}P^n$ be an embedded curve of genus $g > 1$ and let M be the total space of the induced Hopf bundle $f^*(S^1 \times S^{2n+1})$. Then M is Vaisman and $b_1(M) > 1$ (see [V]), hence, according to 6.9, it does not admit any formal Vaisman metric.*

(ii) *On the other hand, we do not have an example of a topologically formal complex compact manifold, which admits Vaisman metrics, but ddoes not admit geometrically formal Vaisman metrics. This seems to be a difficult open problem.*

(iii) *Theorem 6.9 may be considered as an analogue of the following result on the geometric formality of Sasakian manifolds:*

Theorem 6.11 *[GN, Theorem 2.1] Let (M^{2n+1}, g) be a compact Sasakian manifold. If the metric g is geometrically formal, then $b_p(M) = 0$ for $1 \leq p \leq 2n$,* i.e. *M is a real cohomology sphere.*

A Auxiliary results

We collect here some results which are needed in our arguments.

A.1 A characterisation of geometric formality

Lemma 6.12 *Let α and β be two harmonic forms on a compact Riemannian manifold (M^n, g). Then $\alpha \wedge \beta$ is harmonic if and only if the following equality is satisfied:*

$$\sum_{i=1}^{n} (e_i \lrcorner \alpha) \wedge \nabla_{e_i} \beta = -(-1)^{|\alpha||\beta|} \sum_{i=1}^{n} (e_i \lrcorner \beta) \wedge \nabla_{e_i} \alpha, \tag{6.4}$$

where $\{e_i\}_{i=\overline{1,n}}$ is a local orthonormal basis of vector fields. Thus, the metric g is formal if and only if (6.4) *holds for any two g-harmonic forms.*

Proof. Since M is compact, $\alpha \wedge \beta$ is harmonic if and only if it is closed and coclosed. As $\alpha \wedge \beta$ is closed, we have to show that (6.4) is equivalent to $\alpha \wedge \beta$ being coclosed. This is implied by the following:

$$\begin{aligned}
\delta(\alpha \wedge \beta) &= -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} (\alpha \wedge \beta) = -\sum_{i=1}^{n} e_i \lrcorner (\nabla_{e_i} \alpha \wedge \beta + \alpha \wedge \nabla_{e_i} \beta) \\
&= \delta\alpha \wedge \beta - (-1)^{|\alpha|} \sum_{i=1}^{n} \nabla_{e_i} \alpha \wedge (e_i \lrcorner \beta) - \sum_{i=1}^{n} (e_i \lrcorner \alpha) \wedge \nabla_{e_i} \beta + (-1)^{|\alpha|} \alpha \wedge \delta\beta \\
&= -(-1)^{|\alpha||\beta|} \sum_{i=1}^{n} (e_i \lrcorner \beta) \wedge \nabla_{e_i} \alpha - \sum_{i=1}^{n} (e_i \lrcorner \alpha) \wedge \nabla_{e_i} \beta.
\end{aligned}$$

□

A.2 Riemannian products

Let $(M^{n+m}, g) = (M_1^n, g_1) \times (M_2^m, g_2)$. We denote by $\pi_i : M \to M_i$ the natural projections, which are totally geodesic Riemannian submersions.

One may describe the bundle of p-forms on M as follows:

$$\Lambda^p M = \bigoplus_{k=0}^{p} \pi_1^*(\Lambda^k M_1) \otimes \pi_2^*(\Lambda^{p-k} M_2). \tag{6.5}$$

This identification also works for the space of harmonic forms, namely the harmonic forms on (M, g) can be described in terms of the harmonic forms on the factors (M_1, g_1) and (M_2, g_2). To this end let $\mathcal{H}^k(M_i, g_i)$ be the space of harmonic k-forms on M_i and let $b_k(M_i)$ be the Betti numbers of M_i, $i = 1, 2$. Then the following holds:

Lemma 6.13 *Let $\{\alpha_1^k, \ldots, \alpha_{b_k(M_1)}^k\}$ (resp. $\{\beta_1^k, \ldots, \beta_{b_k(M_2)}^k\}$) be a basis of $\mathcal{H}^k(M_1, g_1)$ (resp. $\mathcal{H}^k(M_2, g_2)$). Then the forms:*

$$\{\pi_1^*(\alpha_s^k) \wedge \pi_2^*(\beta_l^{p-k}) | \ 1 \leq s \leq b_k(M_1), 1 \leq l \leq b_{p-k}(M_2), 0 \leq k \leq p\} \tag{6.6}$$

form a basis of the space of $\mathcal{H}^p(M, g)$, for each $0 \leq p \leq m + n$.

For a proof, see [GH, p. 105].

A.3 Warped products

Let (B^n, g_B) and (F^m, g_F) be two Riemannian manifolds and $\varphi > 0$ be a smooth function on B. Then $M = B \times_\varphi F$ denotes the warped product with the metric $g = \pi^*(g_B) + (\varphi \circ \pi)^2 \sigma^*(g_F)$, where $\pi : M \to B$ and $\sigma : M \to F$ are the natural projections.

Let $\{e_i\}_{i=\overline{1,n}}$ be a local orthonormal basis on B and let $\{f_j\}_{j=\overline{1,m}}$ be a local orthonormal basis on F, which we lift to M and thus obtain a local orthonormal basis of M: $\{\tilde{e}_i, \frac{1}{\varphi \circ \pi}\tilde{f}_j\}_{i=\overline{1,n}; j=\overline{1,m}}$.

Consider the following decomposition of the codifferential on M: $\delta = \delta_1 + \delta_2$, where

$$\delta_1 := -\sum_{i=1}^{n} \tilde{e}_i \lrcorner \nabla_{\tilde{e}_i}, \quad \delta_2 := -\frac{1}{(\varphi \circ \pi)^2} \sum_{j=1}^{m} \tilde{f}_j \lrcorner \nabla_{\tilde{f}_j}.$$

We first determine the commutation relations between the pull-back of forms on B and F with δ_1 and δ_2.

Lemma 6.14 *For $\alpha \in \Omega^*(B)$ and $\beta \in \Omega^*(F)$, the following relations hold:*

$$\delta_1(\sigma^*(\beta)) = 0, \tag{6.7}$$

$$\delta_2(\sigma^*(\beta)) = \frac{1}{(\varphi \circ \pi)^2} \sigma^*(\delta^{g_F}(\beta)), \tag{6.8}$$

$$\delta_2(\pi^*(\alpha)) = -\frac{m}{\varphi \circ \pi} \mathrm{grad}(\varphi \circ \pi) \lrcorner \pi^*(\alpha), \tag{6.9}$$

$$\delta_1(\pi^*(\alpha)) = \pi^*(\delta^{g_B}(\alpha)). \tag{6.10}$$

Proof. Let $\beta \in \Omega^{p+1}(F)$. For any tangent vector fields $X_1, \dots, X_p$ to M we obtain:

$$\begin{aligned}
&\delta_1(\sigma^*(\beta))(X_1, \dots, X_p) = -\sum_{i=1}^{n} (\tilde{e}_i \lrcorner \nabla_{\tilde{e}_i}(\sigma^*\beta))(X_1, \dots, X_p) \\
&= -\sum_{i=1}^{n} \tilde{e}_i(\beta(\sigma_*\tilde{e}_i, \sigma_* X_1, \dots, \sigma_* X_p) \circ \sigma) + \sum_{i=1}^{n} \beta(\sigma_*(\nabla_{\tilde{e}_i}\tilde{e}_i), \sigma_* X_1, \dots, \sigma_* X_p) \\
&\quad + \sum_{i=1}^{n} [\beta(\sigma_*\tilde{e}_i, \sigma_*(\nabla_{\tilde{e}_i} X_1), \dots, \sigma_* X_p) + \dots + \beta(\sigma_*\tilde{e}_i, \sigma_* X_1, \dots, \sigma_*(\nabla_{\tilde{e}_i} X_p))] = 0,
\end{aligned}$$

since $\sigma_*\tilde{e}_i = 0$, because $\tilde{e}_i$ is the lift of a vector field on B and also $\sigma_*(\nabla_{\tilde{e}_i}\tilde{e}_i) = \sigma_*(\widetilde{\nabla^{g_B}_{e_i} e_i}) = 0$. This proves (6.7).

The commutation rule (6.8) is shown as follows:

$$\begin{aligned}
&(\varphi \circ \pi)^2 \delta_2(\sigma^*(\beta))(X_1, \dots, X_p) = -\sum_{j=1}^{m} (\tilde{f}_j \lrcorner \nabla_{\tilde{f}_j}(\sigma^*\beta))(X_1, \dots, X_p) \\
&= -\sum_{j=1}^{m} \tilde{f}_j(\beta(\sigma_*\tilde{f}_j, \sigma_* X_1, \dots, \sigma_* X_p) \circ \sigma) \\
&\quad + \sum_{j=1}^{m} \beta(\sigma_*(\nabla_{\tilde{f}_j}\tilde{f}_j), \sigma_* X_1, \dots, \sigma_* X_p) \circ \sigma \\
&\quad + \sum_{j=1}^{m} [\beta(\sigma_*\tilde{f}_j, \sigma_*(\nabla_{\tilde{f}_j} X_1), \dots, \sigma_* X_p) + \dots + \beta(\sigma_*\tilde{f}_j, \sigma_* X_1, \dots, \sigma_*(\nabla_{\tilde{f}_j} X_p))] \circ \sigma \\
&= -\sum_{j=1}^{m} f_j(\beta(f_j, \sigma_* X_1, \dots, \sigma_* X_p)) \circ \sigma \\
&\quad + \sum_{j=1}^{m} \beta(\sigma_*(\widetilde{\nabla^{g_F}_{f_j} f_j} - \frac{g(\tilde{f}_j, \tilde{f}_j)}{\varphi \circ \pi} \operatorname{grad}(\varphi \circ \pi)), \sigma_* X_1, \dots, \sigma_* X_p) \circ \sigma \\
&\quad + \sum_{j=1}^{m} [\beta(f_j, \sigma_*(\nabla_{\tilde{f}_j} X_1), \dots, \sigma_* X_p) + \dots + \beta(f_j, \sigma_* X_1, \dots, \sigma_*(\nabla_{\tilde{f}_j} X_p))] \circ \sigma,
\end{aligned}$$

where we may again assume, without loss of generality, that X_i are lifts of vector fields Z_i on F: $X_i = \tilde{Z}_i$ for $i = 1, \dots, p$. For a tangent vector field Y to B, each of the above terms vanishes, since $\sigma_*(Y) = 0$. We then get:

$$
\begin{aligned}
&(\varphi \circ \pi)^2 \delta_2(\sigma^*(\beta))(X_1, \dots, X_p) = \\
&= -\sum_{j=1}^{m} f_j(\beta(f_j, Z_1, \dots, Z_p)) \circ \sigma + \sum_{j=1}^{m} \beta(\nabla^{g_F}_{f_j} f_j, Z_1, \dots, Z_p) \circ \sigma \\
&\quad + \sum_{j=1}^{m} [\beta(f_j, \nabla^{g_F}_{f_j} Z_1, \dots, \sigma_* X_p) + \dots + \beta(f_j, Z_1, \dots, \nabla^{g_F}_{f_j} Z_p)] \circ \sigma \\
&= \sigma^*(\delta^{g_F}(\beta))(X_1, \dots, X_p),
\end{aligned}
$$

thus proving (6.8).

The relations (6.9) and (6.10) can be obtained by similar computations, which we omit here. □

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